

JOINING AND GLUING SUTURED FLOER HOMOLOGY

RUMEN ZAREV

ABSTRACT. We give a partial characterization of bordered Floer homology in terms of sutured Floer homology. The bordered algebra and modules are direct sums of certain sutured Floer complexes. The algebra multiplication and algebra action correspond to a new gluing map on SFH . It is defined algebraically, and is a special case of a more general “join” map.

In a follow-up paper we show that this gluing map can be identified with the contact cobordism map of Honda-Kazez-Matić. The join map is conjecturally equivalent to the cobordism maps on SFH defined by Juhász.

1. INTRODUCTION

Heegaard Floer homology is a family of invariants for 3 and 4-manifold invariants defined by counting pseudo-holomorphic curves, originally introduced by Ozsváth and Szabó. The most simple form associates to an oriented 3-manifold Y a graded homology group $\widehat{HF}(Y)$ [OS04b, OS04a].

While Heegaard Floer theory for closed 3-manifolds has been very successful, a lot of the applications involve manifolds with boundary. In [Juh06] Juhász introduced sutured Floer homology, or SFH , which generalizes \widehat{HF} to sutured manifolds. Introduced by Gabai in [Gab83], they are 3-manifolds with boundary, and some extra structure. In the context of Heegaard Floer homology, the extra structure can be considered to be a multicurve Γ , called a *dividing set*, on the boundary of the 3-manifold Y . Sutured Floer homology associates to such a pair (Y, Γ) a homology group $SFH(Y, \Gamma)$.

Among other applications, sutured Floer homology has been used to solve problems in contact topology, via a contact invariant for contact manifolds with boundary, and a map associated to contact cobordisms, defined by Honda, Kazez, and Matić in [HKM07, HKM08]. This map has been used by Juhász in [Juh09] to define a map on SFH associated to a cobordism (with corners) between two sutured manifolds.

A shortcoming of sutured Floer homology is that there is little relationship between the groups $SFH(Y, \Gamma_1)$ and $SFH(Y, \Gamma_2)$, where Γ_1 and Γ_2 are two dividing sets on the same manifold Y . For example one can find many examples where one of the groups vanishes, while the other does not. Moreover,

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the groups $SFH(Y_1, \Gamma_1)$ and $SFH(Y_2, \Gamma_2)$ are not sufficient to reconstruct $\widehat{HF}(Y)$, where $Y = Y_1 \cup Y_2$ is a closed manifold.

To overcome these shortcomings, Lipshitz, Ozsváth, and Thurston introduced in [LOT08] a new Heegaard Floer invariant for 3-manifolds with boundary called *bordered Floer homology*. To a parametrized closed connected surface F they associate a DG-algebra $\mathcal{A}(F)$. To a 3-manifold Y with boundary $\partial Y \cong F$ they associate an \mathcal{A}_∞ -module $\widehat{CFA}(Y)$ over $\mathcal{A}(F)$ (defined up to \mathcal{A}_∞ -homotopy equivalence). This invariant overcomes both of the above shortcomings of SFH . On the one hand, given two parametrizations of the surface F , the modules $\widehat{CFA}(Y)$ associated to these parametrizations can be computed from each other. On the other hand, if Y_1 and Y_2 are two manifolds with boundary diffeomorphic to F , the group $\widehat{HF}(Y_1 \cup_F Y_2)$ can be computed from $\widehat{CFA}(Y_1)$ and $\widehat{CFA}(Y_2)$.

The natural question arises: How are these two theories for 3-manifolds with boundary related to each other? Can $SFH(Y, \Gamma)$ be computed from $\widehat{CFA}(Y)$, and if yes, how? Can $\widehat{CFA}(Y)$ be computed from the sutured homology of Y , and if yes, how?

In [Zar09] we introduced *bordered sutured Floer homology*, to serve as a bridge between the two worlds. We used it to answer the first part of the above question—to each dividing set Γ on F we can associate a module $\widehat{CFD}(\Gamma)$ over $\mathcal{A}(F)$, such that $SFH(Y, \Gamma)$ is simply the homology of the derived tensor product $\widehat{CFA}(Y) \otimes \widehat{CFD}(\Gamma)$.

In the current paper we answer the second half of this question. We show that for a given parametrization of F , the homologies of the bordered algebra $\mathcal{A}(F)$ and the module $\widehat{CFA}(Y)$ associated to a 3-manifold Y are direct sums of finitely many sutured Floer homology groups. Moreover we identify multiplication in $H_*(\mathcal{A}(F))$ and the action of $H_*(\mathcal{A}(F))$ on $H_*(\widehat{CFA}(Y))$ with a certain *gluing map* Ψ on sutured Floer homology.

1.1. Results. The first result of this paper is to define the gluing map Ψ discussed above. Suppose (Y_1, Γ_1) and (Y_2, Γ_2) are two sutured manifolds. We say that we can glue them if there are subsets F_1 and F_2 of their boundaries, where F_1 can be identified with the mirror of F_2 , such that the multicurve $\Gamma_1 \cap F_1$ is identified with $\Gamma_2 \cap F_2$, preserving the orientations on Γ_i . This means that the regions R_+ and R_- on the two boundaries are interchanged. We will only talk of gluing in the case when F_i have no closed components, and all components of ∂F_i intersect the dividing sets Γ_i .

Definition 1.1. Suppose (Y_1, Γ_1) , (Y_2, Γ_2) , F_1 and F_2 are as above. The gluing of (Y_1, Γ_1) and (Y_2, Γ_2) along F_i is the sutured manifold $(Y_1 \cup_{F_i} Y_2, \Gamma_{1+2})$. The dividing set Γ_{1+2} is obtained from $(\Gamma_1 \setminus F_1) \cup_{\partial F_i} (\Gamma_2 \setminus F_2)$ as follows. Along each component f of ∂F_i the orientations of Γ_1 and Γ_2 disagree. We apply the minimal possible positive fractional Dehn twist along f that gives a consistent orientation.

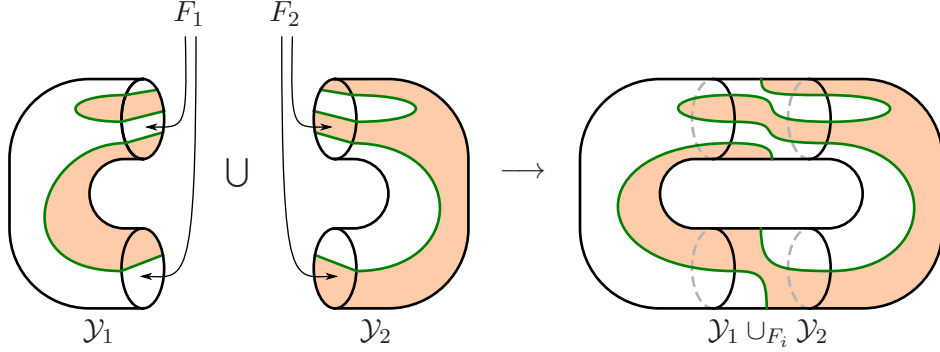


FIGURE 1. Gluing two solid balls along $F = D^2 \cup D^2$, to obtain a solid torus. The R_+ regions have been shaded.

An illustration of gluing is given in Figure 1. We define a gluing map Ψ on SFH corresponding to this topological construction.

Theorem 1. *Let (Y_1, Γ_1) and (Y_2, Γ_2) be two balanced sutured manifolds, that can be glued along F . Then there is a well defined map*

$$\Psi_F: SFH(Y_1, \Gamma_1) \otimes SFH(Y_2, \Gamma_2) \rightarrow SFH((Y_1, \Gamma_1) \cup_F (Y_2, \Gamma_2)),$$

satisfying the following properties:

- (1) *Symmetry:* The map Ψ_F for gluing Y_1 to Y_2 is equal to that for gluing Y_2 to Y_1 .
- (2) *Associativity:* Suppose that we can glue Y_1 to Y_2 along F_1 , and Y_2 to Y_3 along F_2 , such that F_1 and F_2 are disjoint in ∂Y_2 . Then the order of gluing is irrelevant:

$$\Psi_{F_2} \circ \Psi_{F_1} = \Psi_{F_1} \circ \Psi_{F_2} = \Psi_{F_1 \cup F_2}.$$

- (3) *Identity:* Given a dividing set Γ on F , there is a dividing set Γ' on $F \times [0, 1]$, and an element $\Delta_\Gamma \in SFH(F \times [0, 1], \Gamma')$, satisfying the following. For any sutured manifold (Y, Γ'') with $F \subset \partial Y$ and $\Gamma'' \cap F = \Gamma$, there is a diffeomorphism $(Y, \Gamma'') \cup_F (F \times [0, 1], \Gamma') \cong (Y, \Gamma'')$. Moreover, the map $\Psi_F(\cdot, \Delta_\Gamma)$ is the identity of $SFH(Y, \Gamma'')$.

One application of this result is the following characterization of bordered Floer homology in terms of SFH and the gluing map. Fix a parametrized closed surface F , with bordered algebra $A = \mathcal{A}(F)$. Let F' be F with a disc removed, and let $p, q \in \partial F'$ be two points. We can find $2^{2g(F)}$ distinguished dividing sets on F , which we denote Γ_I for $I \subset \{1, \dots, 2g\}$, and corresponding dividing sets $\Gamma'_I = \Gamma_I \cap F'$ on F' . Let $\Gamma_{I \rightarrow J}$ be a dividing set on $F' \times [0, 1]$ which is Γ'_I along $F' \times \{0\}$, Γ'_J along $F' \times \{1\}$, and half of a negative Dehn twist of $\{p, q\} \times [0, 1]$ along $\partial F' \times [0, 1]$.

Theorem 2. *Suppose the surfaces F and F' , the algebra A , and the dividing sets Γ_I , Γ'_I , and $\Gamma_{I \rightarrow J}$ are as described above. Then there is an isomorphism*

$$H_*(A) \cong \bigoplus_{I, J \subset \{1, \dots, 2g\}} SFH(F' \times [0, 1], \Gamma_{I \rightarrow J}),$$

and the multiplication map μ_2 on $H_(A)$ can be identified with the gluing map $\Psi_{F'}$. It maps $SFH(F' \times [0, 1], \Gamma_{I \rightarrow J}) \otimes SFH(F' \times [0, 1], \Gamma_{J \rightarrow K})$ to $SFH(F' \times [0, 1], \Gamma_{I \rightarrow K})$, and sends all other summands to 0.*

The module \widehat{CFA} can be similarly described.

Theorem 3. *Suppose Y is a 3-manifold with boundary $\partial Y \cong F$. There is an isomorphism*

$$H_*(\widehat{CFA}(Y)_A) \cong \bigoplus_{I \subset \{1, \dots, 2g\}} SFH(Y, \Gamma_I),$$

and the action m_2 of $H_(A)$ on $H_*(\widehat{CFA}(Y))$ can be identified with the gluing map $\Psi_{F'}$. It maps $SFH(Y, \Gamma_I) \otimes SFH(F' \times [0, 1], \Gamma_{I \rightarrow J})$ to $SFH(Y, \Gamma_J)$, and sends all other summands to 0.*

The gluing construction and the gluing map readily generalize to a more general *join* construction, and *join map*, which are 3-dimensional analogs. Suppose that (Y_1, Γ_1) and (Y_2, Γ_2) are two sutured manifolds, and F_1 and F_2 are subsets of their boundaries, satisfying the conditions for gluing. Suppose further that the diffeomorphism $F_1 \rightarrow F_2$ extends to $W_1 \rightarrow W_2$, where W_i is a compact codimension-0 submanifold of Y_i , and $\partial W_i \cap \partial Y_i = F_i$. Instead of gluing Y_1 and Y_2 along F_i , we can *join* them along W_i .

Definition 1.2. *The join of (Y_1, Γ_1) and (Y_2, Γ_2) along W_i is the sutured manifold*

$$((Y_1 \setminus W_1) \cup_{\partial W_i \setminus F_i} (Y_2 \setminus W_2), \Gamma_{1+2}),$$

where the dividing set Γ_{1+2} is constructed exactly as in Definition 1.1. We denote the join by $(Y_1, \Gamma_1) \mathbb{U}_{W_i} (Y_2, \Gamma_2)$.

An example of a join is shown in Figure 2. Notice that if W_i is a collar neighborhood of F_i , then the notions of join and gluing coincide. That is, the join operation is indeed a generalization of gluing. In fact, throughout the paper we work almost exclusively with joins, while only regarding gluing as a special case.

Theorem 4. *There is a well-defined join map*

$$\Psi_W: SFH(Y_1, \Gamma_1) \otimes SFH(Y_2, \Gamma_2) \rightarrow SFH((Y_1, \Gamma_1) \mathbb{U}_W (Y_2, \Gamma_2)),$$

satisfying properties of symmetry, associativity, and identity, analogous to those listed in Theorem 1.

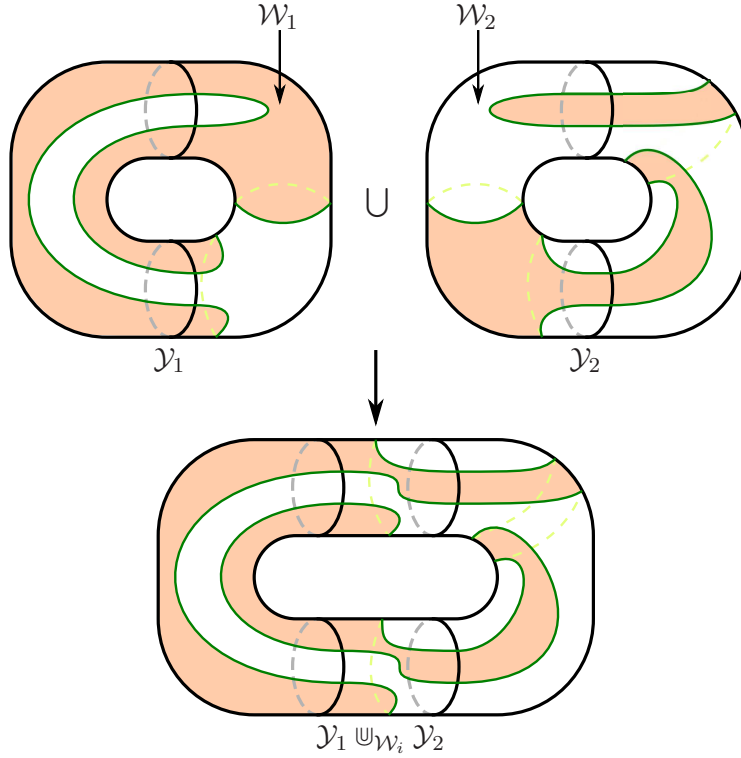


FIGURE 2. Join of two solid tori along $D^2 \times [0, 1]$, to obtain another solid torus. The R_+ regions have been shaded.

The join map is constructed as follows. We cut out W_1 and W_2 from Y_1 and Y_2 , respectively, and regard the complements as bordered sutured manifolds. The join operation corresponds to replacing W_1 and W_2 by an interpolating piece $\mathcal{TW}_{F,+}$. We define a map between the bordered sutured invariants, from the product $\widehat{BSA}(W_1) \otimes \widehat{BSA}(W_2)$ to the bimodule $\widehat{BSAA}(\mathcal{TW}_{F,+})$. We show that for an appropriate choice of parametrizations, the modules $\widehat{BSA}(W_1)$ and $\widehat{BSA}(W_2)$ are duals, while $\widehat{BSAA}(\mathcal{TW}_{F,+})$ is the dual of the bordered algebra for F . The map is then an \mathcal{A}_∞ -version of the natural pairing between a module and its dual. The proof of invariance and the properties from Theorems 1 and 4 is purely algebraic. Most of the arguments involve \mathcal{A}_∞ -versions of standard facts in commutative algebra.

The proofs of Theorems 2 and 3 involve several steps. First, we find a manifold whose bordered sutured invariant is the bordered algebra, as a bimodule over itself. Second, we find manifolds whose bordered sutured invariants are all possible simple modules over the algebra. Finally, we compute the gluing map Ψ explicitly in several cases.

1.2. Further implications and speculations. In a follow-up paper [Zar] we prove that the gluing map map Ψ_F can be identified with the contact

cobordism map from [HKM08]. This is somewhat surprising as that the definition of that map uses contact structures and open books, while our definition uses bordered sutured Floer homology and is purely algebraic. The equivalence of the two maps also gives a purely contact-geometric interpretation of the bordered algebra.

There is no analog of the join map in the setting of Honda, Kazez, and Matić. However, there is a natural pair-of-pants cobordism

$$Z_W: (Y_1, \Gamma_1) \sqcup (Y_2, \Gamma_2) \rightarrow (Y_1, \Gamma_1) \mathbin{\mathbb{U}}_W (Y_2, \Gamma_2),$$

and we conjecture that the join map Ψ_W is equivalent to the cobordism map F_{Z_W} that Juhász associates to such a cobordism, by counting pseudo-holomorphic triangles.

Though Theorems 2 and 3 give a pretty good description of bordered Floer homology in terms of sutured Floer homology, it is not complete. For instance, to be able to recover the pairing theorem for bordered Floer homology, we need to work either with the full bordered DG-algebra $\mathcal{A}(F)$, or with its homology $H_*(\mathcal{A}(F))$, considered as an \mathcal{A}_∞ -algebra. That is, $H_*(\mathcal{A}(F))$ inherits higher multiplication maps μ_i , for $i \geq 2$ from the DG-structure on $\mathcal{A}(F)$. Theorem 2 only recovers μ_2 . Similarly, $H_*(\widehat{CFA}(Y))$ has higher actions m_i , for $i \geq 2$ by $H_*(\mathcal{A}(F))$, while Theorem 3 only recovers m_2 .

We believe that these higher structures can be recovered by a similar construction. There are maps, $\overline{\Psi}_i$, for $i \geq 2$, defined algebraically, similar to Ψ , of the following form:

$$\overline{\Psi}_i: SFC(Y_1) \otimes \cdots \otimes SFC(Y_i) \rightarrow SFC(Y_1 \mathbin{\mathbb{U}} \cdots \mathbin{\mathbb{U}} Y_i).$$

Here SFC denotes the chain complex defining the homology group SFH . The first term $\overline{\Psi}_2$ induces the usual join Ψ on homology.

Conjecture 5. *The following two statements hold:*

- (1) *The collection of maps $\overline{\Psi}_i$, for $i \geq 2$ can be used to recover the higher multiplications μ_i on $H_*(\mathcal{A}(F))$, and the higher actions m_i of $H_*(\mathcal{A}(F))$ on $H_*(\widehat{CFA}(Y))$.*
- (2) *The map $\overline{\Psi}_i$ can be computed by counting pseudo-holomorphic $(i+1)$ -gons in a sutured Heegaard multidiagram.*

Analogues of sutured Floer homology have been defined in settings other than Heegaard Floer homology—for instanton and monopole Floer homology in [KM10], and for embedded contact homology in [CGHH10]. We believe that analogues of the join and gluing maps can be used to extend bordered Floer homology to those settings.

Organization. We start by introducing in more detail the topological constructions of the gluing join operations in Section 2. In Section 3 we recall briefly the definitions of the bordered sutured invariants \widehat{BSA} and \widehat{BSD} . We

also discuss how the original definitions involving only α -arcs can be extended to diagrams using both β -arcs, and to some mixed diagrams using both. Section 3.4 contains computations of several \widehat{BSA} invariants needed later.

We define the join map in Section 4, on the level of chain complexes. The same section contains the proof that it descends to a unique map on homology. In the following Section 5 we prove the properties from Theorems 1 and 4. Finally, Section 6 contains the statement and the proof of a slightly more general version of Theorems 2 and 3.

Throughout the paper, we make use of a diagrammatic calculus to compute \mathcal{A}_∞ -morphisms, which greatly simplifies the arguments. Appendix A contains a brief description of this calculus, and the necessary algebraic assumptions. Appendix B gives an overview of \mathcal{A}_∞ -bimodules in terms of the diagrammatic calculus, as they are used in the paper.

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2. TOPOLOGICAL PRELIMINARIES

We recall the definition of a sutured manifold and some auxiliary notions, and define what we mean by gluing and surgery.

Remark. Throughout the paper all manifolds are oriented. We use $-M$ to denote the manifold M with its orientation reversed.

2.1. Sutured manifolds and surfaces.

Definition 2.1. *As defined in [Juh06], a balanced sutured manifold is a pair $\mathcal{Y} = (Y, \Gamma)$ consisting of the following:*

- *An oriented 3-manifold Y with boundary.*
- *A collection Γ of disjoint oriented simple closed curves in ∂Y , called sutures.*

They are required to satisfy the following conditions:

- *Y can be disconnected but cannot have any closed components.*
- *∂Y is divided by Γ into two complementary regions $R_+(\Gamma)$ and $R_-(\Gamma)$ such that $\partial R_\pm(Y) = \pm\Gamma$. (R_+ and R_- may be disconnected.)*
- *Each component of ∂Y contains a suture. Equivalently, R_+ and R_- have no closed components.*
- *$\chi(R_+) = \chi(R_-)$.*

In [Zar09] we introduced the notion of a sutured surface.

Definition 2.2. A *sutured surface* is a pair $\mathcal{F} = (F, \Lambda)$ consisting of the following:

- A compact oriented surface F .
- A finite collection $\Lambda \subset \partial F$ of points with sign, called sutures.

They are required to satisfy the following conditions:

- F can be disconnected but cannot have any closed components.
- ∂F is divided by Λ into two complementary regions $S_+(\Gamma)$ and $S_-(\Gamma)$ such that $\partial S_\pm(Y) = \pm\Lambda$. (S_+ and S_- may be disconnected.)
- Each component of ∂F contains a suture. Equivalently, S_+ and S_- have no closed components.

A sutured surface is precisely the 2-dimensional equivalent of a balanced sutured manifold. The requirement $\chi(S_+) = \chi(S_-)$ follows automatically from the other conditions.

From $\mathcal{F} = (F, \Lambda)$ we can construct two other sutured surfaces: $-\mathcal{F} = (-F, -\Lambda)$, and $\overline{\mathcal{F}} = (-F, \Lambda)$. In both of $-\mathcal{F}$ and $\overline{\mathcal{F}}$, the orientation of the underlying surface F is reversed. The difference between the two is that in $-\mathcal{F}$ the roles of S_+ and S_- are preserved, while in $\overline{\mathcal{F}}$ they are reversed.

Definition 2.3. Suppose $\mathcal{F} = (F, \Lambda)$ is a sutured surface. A *dividing set* Γ for \mathcal{F} is a finite collection Γ of disjoint embedded oriented arcs and simple closed curves in F , with the following properties:

- $\partial\Gamma = -\Lambda$, as an oriented boundary.
- Γ divides F into (possibly disconnected) regions R_+ and R_- with $\partial R_\pm = (\pm\Gamma) \cup S_\pm$.

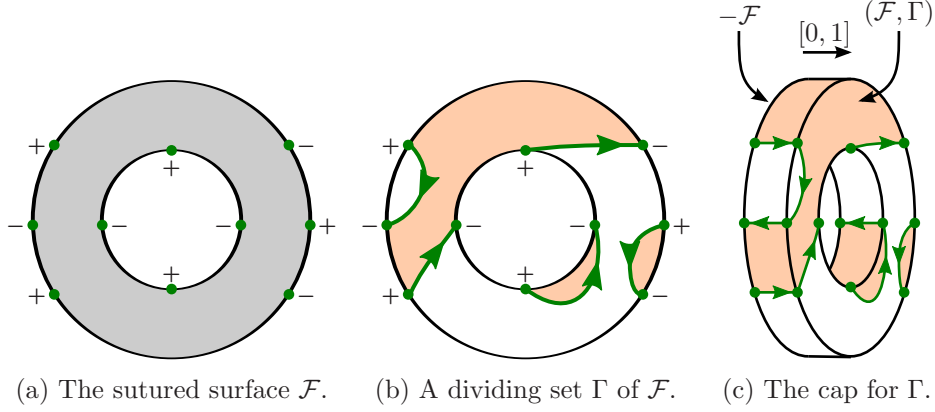
We can extend the definition of a dividing set to pairs (F, Λ) which do not quite satisfy the conditions for a sutured surface. We can allow some or all of the components F to be closed. We call such a pair *degenerate*. In that case we impose the extra condition that each closed component contains a component of Γ .

Note that the sutures Γ of a sutured manifold (Y, Γ) can be regarded as a dividing set for the (degenerate) sutured surface $(\partial Y, \emptyset)$.

Definition 2.4. A *partially sutured manifold* is a triple $\mathcal{Y} = (Y, \Gamma, \mathcal{F})$ consisting of the following:

- A 3-manifold Y with boundary and 1-dimensional corners.
- A sutured surface $\mathcal{F} = (F, \Lambda)$, such that $F \subset \partial Y$, and such that the 1-dimensional corner of Y is ∂F .
- A dividing set Γ for $(\partial Y \setminus F, -\Lambda)$ (which might be degenerate).

Note that a partially sutured manifold $\mathcal{Y} = (Y, \Gamma, \mathcal{F}_1 \sqcup \mathcal{F}_2)$ can be thought of as a cobordism between $-\mathcal{F}_1$ and \mathcal{F}_2 . On the other hand, the partially sutured manifold $\mathcal{Y} = (Y, \Gamma, \emptyset)$ is just a sutured manifold, although it may not be balanced. We can *concatenate* $\mathcal{Y} = (Y, \Gamma, \mathcal{F}_1 \sqcup \mathcal{F}_2)$ and $\mathcal{Y}' = (Y', \Gamma', -\mathcal{F}_2 \sqcup \mathcal{F}_3)$ along $\mathcal{F}_2 = (F_2, \Lambda_2)$ and $-\mathcal{F}_2 = (-F_2, -\Lambda_2)$ to

FIGURE 3. A sutured annulus \mathcal{F} , with a cap associated to a dividing set.

obtain

$$\mathcal{Y} \cup_{\mathcal{F}_2} \mathcal{Y}' = (Y \cup_{F_2} Y', \Gamma \cup_{\Lambda_2} \Gamma', \mathcal{F}_1 \sqcup \mathcal{F}_3).$$

We use the term concatenate to distinguish from the operation of *gluing* of two sutured manifolds described in Definition 2.10.

A partially sutured manifold whose sutured surface is parametrized by an arc diagram is a *bordered sutured manifold*, as defined in [Zar09]. We will return to this point in section 3, where we give the precise definitions.

An important special case is when Y is a thickening of F .

Definition 2.5. Suppose Γ is a dividing set for the sutured surface $\mathcal{F} = (F, \Lambda)$. Let $W = F \times [0, 1]$, and $W' = F \times [0, 1] / \sim$, where $(p, t) \sim (p, t')$ whenever $p \in \partial F$, and $t, t' \in [0, 1]$. We will refer to the partially sutured manifolds

$$\begin{aligned} \mathcal{W}_\Gamma &= (W, \Gamma \times \{1\} \cup \Lambda \times [0, 1], (-F \times \{0\}, -\Lambda \times \{0\})), \\ \mathcal{W}'_\Gamma &= (W', \Gamma \times \{1\}, (-F \times \{0\}, -\Lambda \times \{0\})) \end{aligned}$$

as the caps for \mathcal{F} associated to Γ .

Since \mathcal{W}'_Γ is just a smoothing of \mathcal{W}_Γ along the corner $\partial F \times \{1\}$, we will not distinguish between them. An illustration of a dividing set and a cap is shown in Figure 3. In this and in all other figures we use the convention that the dividing set is colored in green, to avoid confusion with Heegaard diagrams later. We also shade the R_+ regions.

Notice that the sutured surface for \mathcal{W}_Γ is $-\mathcal{F}$. This means that if $\mathcal{Y} = (Y, \Gamma', \mathcal{F})$ is a partially sutured manifold, we can concatenate \mathcal{Y} and \mathcal{W} to obtain $(Y, \Gamma' \cup \Gamma)$. That is, the effect is that of “filling in” $F \subset \partial Y$ by Γ .

Definition 2.6. Suppose $\mathcal{F} = (F, \Lambda)$ is a sutured surface. An embedding $\mathcal{W} \hookrightarrow \mathcal{Y}$ of the partially sutured $\mathcal{W} = (W, \Gamma_W, \mathcal{F})$ into the sutured $\mathcal{Y} = (Y, \Gamma_Y)$ is an embedding $W \hookrightarrow Y$ with the following properties:

- $F \subset \partial W$ is properly embedded in Y as a separating surface.

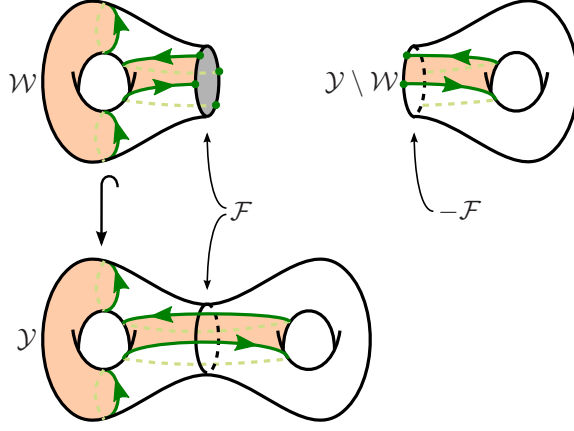


FIGURE 4. Examples of a partially sutured manifold \mathcal{W} embedding into the sutured manifold \mathcal{Y} , and the complement $\mathcal{Y} \setminus \mathcal{W}$, which is also partially sutured.

- $\partial W \setminus F = \partial Y \cap W$.
- $\Gamma_W = \Gamma_Y \cap \partial W$.

The complement $Y \setminus W$ also inherits a partially sutured structure. We define

$$\mathcal{Y} \setminus \mathcal{W} = (Y \setminus W, \Gamma_Y \setminus \Gamma_W, -\mathcal{F}).$$

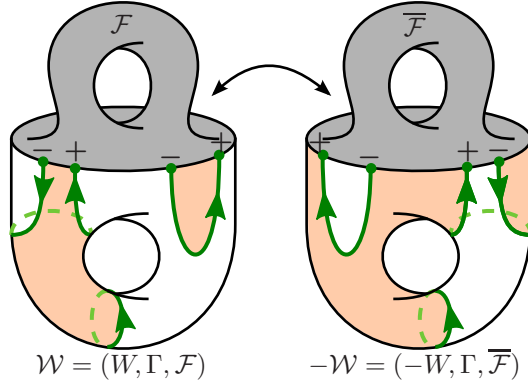
The definition of embeddings easily extends to $\mathcal{W} \hookrightarrow \mathcal{Y}$ where both $\mathcal{W} = (W, \Gamma_W, \mathcal{F})$ and $\mathcal{Y} = (Y, \Gamma_Y, \mathcal{F}')$ are partially sutured. In this case we require that W is disjoint from a collar neighborhood of F' . Then there is still a complement

$$\mathcal{Y} \setminus \mathcal{W} = (Y \setminus W, \Gamma_Y \setminus \Gamma_W, \mathcal{F}' \cup -\mathcal{F}).$$

In both cases \mathcal{Y} is diffeomorphic to the concatenation $\mathcal{W} \cup_{\mathcal{F}} (\mathcal{Y} \setminus \mathcal{W})$. Examples of a partial sutured manifold and of an embedding are given in Figure 4.

2.2. Mirrors and doubles; joining and gluing. We want to define a gluing operation which takes two sutured manifolds (Y_1, Γ_1) and (Y_2, Γ_2) , and surfaces $F \subset \partial Y_1$ and $-F \subset \partial Y_2$, and produces a new sutured manifold $(Y_1 \cup_F Y_2, \Gamma_3)$. To do that we have to decide how to match up the dividing sets on and around F and $-F$. One solution is to require that we glue $F \cap R_+(\Gamma_1)$ to $-F \cap R_+(\Gamma_2)$, and $F \cap R_-(\Gamma_1)$ to $-F \cap R_-(\Gamma_2)$. Then $(\Gamma_1 \setminus F) \cup (\Gamma_2 \setminus -F)$ is a valid dividing set, and candidate for Γ_3 . The problem with this approach is that even if we glue two balanced sutured manifolds, the result is not guaranteed to be balanced.

Another approach, suggested by contact topology is the following. We glue $F \cap R_+$ to $-F \cap R_-$, and vice versa. To compensate for the fact that the dividing sets $\Gamma_1 \setminus F$ and $\Gamma_2 \setminus -F$ do not match up anymore, we introduce a slight twist along ∂F . In contact topology this twist appears when we smooth the corner between two convex surfaces meeting at an angle.

FIGURE 5. A partially sutured manifold \mathcal{W} and its mirror $-\mathcal{W}$.

It turns out that the same approach is the correct one, from the bordered sutured point of view. To be able to define a gluing map on SFH with nice formal properties, the underlying topological operation should employ the same kind of twist. However, its direction is opposite from the one in the contact world. This is not unexpected, as orientation reversal is the norm when defining any contact invariant in Heegaard Floer homology.

As we briefly explained in Section 1, we will also define a surgery procedure which we call *joining*, and which generalizes this gluing operation. We will associate a map on sutured Floer homology to such a surgery in Section 4.2.

First we define some preliminary notions.

Definition 2.7. *The mirror of a partially sutured manifold $\mathcal{W} = (W, \Gamma, \mathcal{F})$, where $\mathcal{F} = (F, \Lambda)$ is $-\mathcal{W} = (-W, \Gamma, \overline{\mathcal{F}})$. Alternatively, it is a partially sutured manifold $(W', \Gamma', \mathcal{F}')$, with an orientation reversing diffeomorphism $\varphi: W \rightarrow W'$, such that:*

- F is sent to $-F'$ (orientation is reversed).
- Γ is sent to Γ' (orientation is preserved).
- $R_+(\Gamma)$ is sent to $R_-(\Gamma')$, and vice versa.
- $S_+(\Lambda)$ is sent to $S_-(\Lambda')$, and vice versa.

Whenever we talk about a pair of mirrors, we will implicitly assume that a specific diffeomorphism between them has been chosen. An example is shown in Figure 5.

There are two partially sutured manifolds, which will play an important role.

Definition 2.8. *A positive (respectively negative) twisting slice along the sutured surface $\mathcal{F} = (F, \Lambda)$ is the partially sutured manifold $\mathcal{TW}_{\mathcal{F}, \pm} = (F \times [0, 1], \Gamma, -\mathcal{F} \cup -\overline{\mathcal{F}})$ where we identify $-\mathcal{F}$ with $F \times \{0\}$, and $-\overline{\mathcal{F}}$ with $F \times \{1\}$. The dividing set Γ is obtained from $\Lambda \times [0, 1]$ by applying $\frac{1}{n}$ -th of a positive (respectively negative) Dehn twist along each component of*

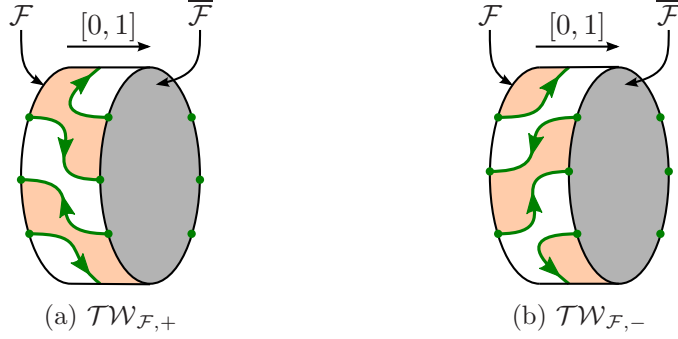


FIGURE 6. Positive and negative twisting slices. The dividing sets are $\Lambda \times [0, 1]$, after a fractional Dehn twist has been applied. The R_+ regions have been shaded.

$\partial F \times \{\frac{1}{2}\}$, containing n points of Λ . (The twists might be different for different components.)

Examples of twisting slices are shown in Figure 6.

Definition 2.9. Let \mathcal{Y}_1 and \mathcal{Y}_2 be sutured manifolds, and $\mathcal{W} = (W, \Gamma, -\mathcal{F})$ be partially sutured. Suppose there are embeddings $\mathcal{W} \hookrightarrow \mathcal{Y}_1$ and $-\mathcal{W} \hookrightarrow \mathcal{Y}_2$. We will call the new sutured manifold

$$\mathcal{Y}_1 \mathbin{\mathbb{W}}_{\mathcal{W}} \mathcal{Y}_2 = (\mathcal{Y}_1 \setminus \mathcal{W}) \cup_{\mathcal{F}} \mathcal{TW}_{\mathcal{F},+} \cup_{-\mathcal{F}} (\mathcal{Y}_2 \setminus -\mathcal{W})$$

the join of \mathcal{Y}_1 and \mathcal{Y}_2 along \mathcal{W} .

Intuitively, this means that we cut out \mathcal{W} and $-\mathcal{W}$ and concatenate the complements together. There is a mismatch of R_+ with R_- along the boundary, so we introduce a positive twist to fix it. An example of gluing was shown in Figure 2.

Another important operation is gluing.

Definition 2.10. Suppose that $\mathcal{Y}_1 = (Y_1, \Gamma_1, \mathcal{F})$ and $\mathcal{Y}_2 = (Y_2, \Gamma_2, \overline{\mathcal{F}})$ are two partially sutured manifolds, and Γ_0 is a dividing set for $\mathcal{F} = (F, \Lambda)$. We define the gluing of the sutured manifolds $(Y_1, \Gamma_1 \cup_{\Lambda} \Gamma_0)$ and $(Y_2, \Gamma_2 \cup_{\Lambda} \Gamma_0)$ along (F, Γ_0) to be the concatenation

$$\mathcal{Y}_1 \cup_{-\mathcal{F}} \mathcal{TW}_{\mathcal{F},+} \cup_{\overline{\mathcal{F}}} \mathcal{Y}_2,$$

and denote it by

$$(Y_1, \Gamma_1 \cup \Gamma_0) \cup_{(F, \Gamma_0)} (Y_2, \Gamma_2 \cup \Gamma_0).$$

An example of gluing was shown in Figure 1. It is easy to see that gluing is a special case of the join. Recall that the concatenation $(Y, \Gamma', \mathcal{F}) \cup_{\mathcal{F}} \mathcal{W}_{\Gamma}$ is the sutured manifold $(Y, \Gamma' \cup \Gamma)$. Thus we can identify gluing along (F, Γ_0) with join along \mathcal{W}_{Γ_0} .

Another useful object is the double of a partially sutured manifold.

Definition 2.11. *Given a partially sutured manifold $\mathcal{W} = (W, \Gamma, \mathcal{F})$, where $\mathcal{F} = (F, \Lambda)$, define the double of \mathcal{W} to be the sutured manifold obtained by concatenation as follows:*

$$D(\mathcal{W}) = -\mathcal{W} \cup_{-\overline{\mathcal{F}}} \mathcal{T}\mathcal{W}_{-\overline{\mathcal{F}}, -} \cup_{\mathcal{F}} \mathcal{W}.$$

All the operations we have defined so far keep us in the realm of balanced sutured manifolds.

Proposition 2.12. *If we join or glue two balanced sutured manifolds together, the result is balanced. The double of any partially sutured manifold \mathcal{W} is balanced.*

Proof. There are three key observations. The first one is that $\chi(R_+) - \chi(R_-)$ is additive under concatenation. The second is that when passing from \mathcal{W} to its mirror $-\mathcal{W}$, the values of $\chi(R_+)$ and $\chi(R_-)$ are interchanged. Finally, for positive and negative twisting slices $\chi(R_+) = \chi(R_-)$. \square

The operations of joining and gluing sutured manifolds have good formal properties described in the following proposition.

Proposition 2.13. *The join satisfies the following:*

- (1) *Commutativity: $\mathcal{Y}_1 \mathbin{\mathbb{U}}_{\mathcal{W}} \mathcal{Y}_2$ is canonically diffeomorphic to $\mathcal{Y}_2 \mathbin{\mathbb{U}}_{-\mathcal{W}} \mathcal{Y}_1$.*
- (2) *Associativity: If there are embeddings $\mathcal{W} \hookrightarrow \mathcal{Y}_1$, $(-\mathcal{W} \sqcup \mathcal{W}') \hookrightarrow \mathcal{Y}_2$, and $-\mathcal{W}' \hookrightarrow \mathcal{W}_3$ then there are canonical diffeomorphisms*

$$\begin{aligned} (\mathcal{Y}_1 \mathbin{\mathbb{U}}_{\mathcal{W}} \mathcal{Y}_2) \mathbin{\mathbb{U}}_{\mathcal{W}'} \mathcal{Y}_3 &\cong \mathcal{Y}_1 \mathbin{\mathbb{U}}_{\mathcal{W}} (\mathcal{Y}_2 \mathbin{\mathbb{U}}_{\mathcal{W}'} \mathcal{Y}_3) \\ &\cong (\mathcal{Y}_1 \sqcup \mathcal{Y}_3) \mathbin{\mathbb{U}}_{\mathcal{W} \cup -\mathcal{W}'} \mathcal{Y}_2. \end{aligned}$$

- (3) *Identity: $\mathcal{Y} \mathbin{\mathbb{U}}_{\mathcal{W}} D(\mathcal{W}) \cong \mathcal{Y}$.*

Gluing satisfies analogous properties.

Proof. These facts follow immediately from the definitions. \square

3. BORDERED SUTURED FLOER HOMOLOGY

We recall the definitions of bordered sutured manifolds and their invariants, as introduced in [Zar09].

3.1. Arc diagrams and bordered sutured manifolds. Parametrizations by arc diagrams, as described below are a slight generalization of those originally defined in [Zar09]. The latter corresponded to parametrizations using only α -arcs. While this is sufficient to define invariants for all possible situations, it is somewhat restrictive computationally. Indeed, to define the join map Ψ we need to exploit some symmetries that are not apparent unless we also allow parametrizations using β -arcs.

Definition 3.1. *An arc diagram of rank k is a triple $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ consisting of the following:*

- *A finite collection \mathbf{Z} of oriented arcs.*

- A collection of points $\mathbf{a} = \{a_1, \dots, a_{2k}\} \subset \mathbf{Z}$.
- A 2-to-1 matching $M: \mathbf{a} \rightarrow \{1, \dots, k\}$ of the points into pairs.
- A type: “ α ” or “ β ”.

We require that the 1-manifold obtained by performing surgery on all the 0-spheres $M^{-1}(i)$ in \mathbf{Z} has no closed components.

We represent arc diagrams graphically by a graph $G(\mathcal{Z})$, which consists of the arcs \mathbf{Z} , oriented upwards, and an arc e_i attached at the pair $M^{-1}(i) \in \mathbf{Z}$, for $i = 1, \dots, k$. Depending on whether the diagram is of α or β type, we draw the arcs to the right or to the left, respectively.

Definition 3.2. The sutured surface $\mathcal{F}(\mathcal{Z}) = (F(\mathcal{Z}), \Lambda(\mathcal{Z}))$ associated to the α -arc diagram \mathcal{Z} is constructed in the following way. The underlying surface F is produced from the product $\mathbf{Z} \times [0, 1]$ by attaching 1-handles along the 0-spheres $M^{-1}(i) \times \{0\}$, for $i = 1, \dots, k$. The sutures are $\Lambda = \partial\mathbf{Z} \times \{1/2\}$, with the positive region S_+ being “above”, i.e. containing $\mathbf{Z} \times \{1\}$.

The sutured surface associated to a β -arc diagram is constructed in the same fashion, except that the 1-handles are attached “on top”, i.e. at $M^{-1}(i) \times \{1\}$. The positive region S_+ is still above.

Suppose F is a surface with boundary, $G(\mathcal{Z})$ is properly embedded in F , and $\Lambda = \partial G(\mathcal{Z}) \subset \partial F$ are the vertices of valence 1. If F deformation retracts onto $G(\mathcal{Z})$, we can identify (F, Λ) with $\mathcal{F}(\mathcal{Z})$. In fact, the embedding uniquely determines such an identification, up to isotopies fixing the boundary. We say that \mathcal{Z} parametrizes (F, Λ) .

As mentioned earlier, all arc diagrams considered in [Zar09] are of α -type.

Let $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ be an arc diagram. We will denote by $-\mathcal{Z}$ the diagram obtained by reversing the orientation of \mathbf{Z} (and preserving the type). We will denote by $\overline{\mathcal{Z}}$ the diagram obtained by switching the type—from α to β , or vice versa—and preserving the triple $(\mathbf{Z}, \mathbf{a}, M)$. There are now four related diagrams: \mathcal{Z} , $-\mathcal{Z}$, $\overline{\mathcal{Z}}$, and $-\overline{\mathcal{Z}}$. The notation is intentionally similar to the one for the variations on a sutured surface. Indeed, they are related as follows:

$$\mathcal{F}(-\mathcal{Z}) = -\mathcal{F}(\mathcal{Z}), \quad \mathcal{F}(\overline{\mathcal{Z}}) = \overline{\mathcal{F}(\mathcal{Z})}.$$

To illustrate that, Figure 7 has four variations of an arc diagram of rank 3. Figure 8 shows the corresponding parametrizations of sutured surfaces, which are all tori with one boundary component and four sutures. Notice the embedding of the graph in each case.

Definition 3.3. A bordered sutured manifold $\mathcal{Y} = (Y, \Gamma, \mathcal{Z})$ is a partially sutured manifold (Y, Γ, \mathcal{F}) , whose sutured surface \mathcal{F} has been parametrized by the arc diagram \mathcal{Z} .

As with partially sutured manifolds, $\mathcal{Y} = (Y, \Gamma, \mathcal{Z}_1 \sqcup \mathcal{Z}_2)$ can be thought of as a cobordism from $\mathcal{F}(-\mathcal{Z}_1)$ to $\mathcal{F}(\mathcal{Z}_2)$.

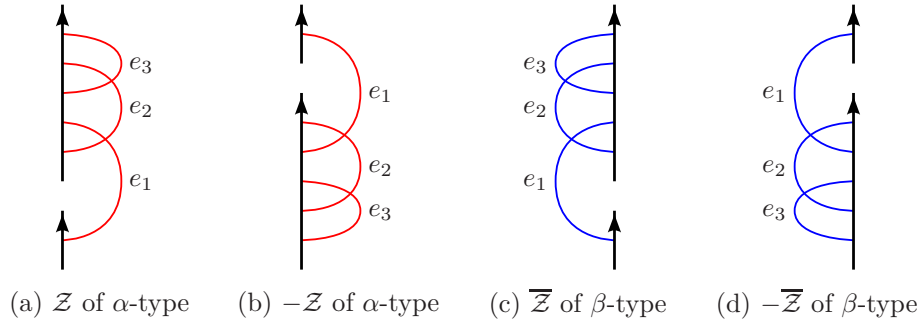


FIGURE 7. Four variants of an arc diagram

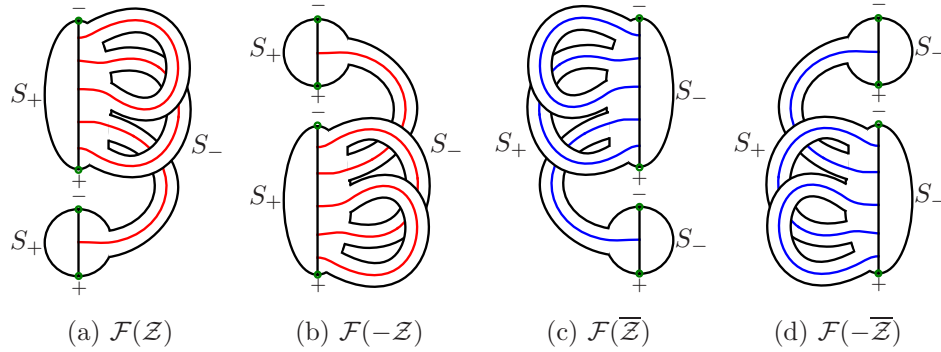


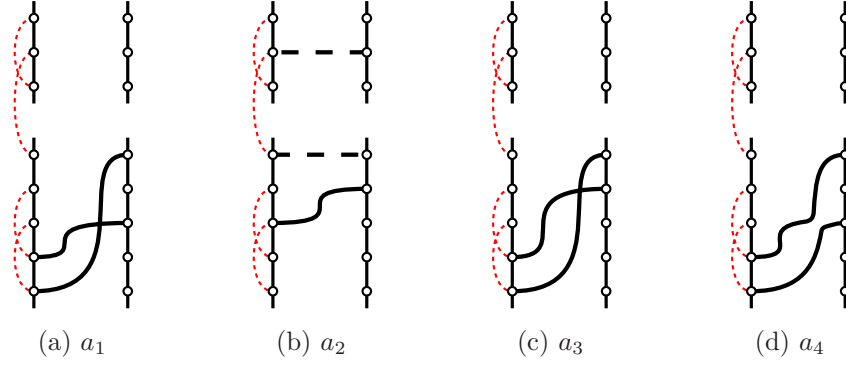
FIGURE 8. Parametrizations of surfaces by the arc diagrams in Figure 7

3.2. The bordered algebra. We will briefly recall the definition of the algebra $\mathcal{A}(\mathcal{Z})$ associated to an α -type arc diagram \mathcal{Z} . Fix a diagram $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ of rank k . First, we define a larger *strands algebra* $\mathcal{A}'(\mathbf{Z}, \mathbf{a})$, which is independent of the matching M . Then we define $\mathcal{A}(\mathcal{Z})$ as a subalgebra of $\mathcal{A}'(\mathbf{Z}, \mathbf{a})$.

Definition 3.4. *The strands algebra associated to (\mathbf{Z}, \mathbf{a}) is a $\mathbb{Z}/2$ -algebra $\mathcal{A}'(\mathbf{Z}, \mathbf{a})$, which is generated (as a vector space) by diagrams in $[0, 1] \times \mathbf{Z}$ of the following type. Each diagram consists of several embedded oriented arcs or strands, starting in $\{0\} \times \mathbf{a}$ and ending in $\{1\} \times \mathbf{a}$. All tangent vectors on the strands should project non-negatively on \mathbf{Z} , i.e. they are “upward-veering”. Only transverse intersections are allowed.*

The diagrams are subjects to two relations—any two diagrams related by a Reidemeister III move represent the same element in $\mathcal{A}'(\mathbf{Z}, \mathbf{a})$, and any diagram in which two strands intersect more than once represents zero.

Multiplication is given by concatenation of diagrams in the $[0, 1]$ -direction, provided the endpoints of the strands agree. Otherwise the product is zero. The differential of a diagram is the sum of all diagrams obtained from it by taking the oriented resolution of a crossing.

FIGURE 9. Four generators of $\mathcal{A}(\mathcal{Z})$.

We refer to a strand connecting $(0, a)$ to $(1, a)$ for some $a \in \mathbf{a}$ as *horizontal*. Notice that the idempotent elements of $\mathcal{A}'(\mathbf{Z}, \mathbf{a})$ are precisely those which are sums of diagrams with only horizontal strands. To recover the information carried by the matching M we single out some of these idempotents.

Definition 3.5. The ground ring $\mathcal{I}(\mathcal{Z})$ associated to \mathcal{Z} is a ground ring, in the sense of Definition A.1, of rank 2^k over $\mathbb{Z}/2$, with canonical basis $(\iota_I)_{I \subset \{1, \dots, k\}}$. It is identified with a subring of the strands algebra $\mathcal{A}'(\mathbf{Z}, \mathbf{a})$, by setting $\iota_I = \sum_J D_J$. The sum is over all $J \subset \mathbf{a}$ such that $M|_J: J \rightarrow I$ is a bijection, and D_J is the diagram with horizontal strands $[0, 1] \times J$.

For all $I \subset \{1, \dots, k\}$, the generator ι_I is a sum of $2^{\#I}$ diagrams.

Definition 3.6. The bordered algebra $\mathcal{A}(\mathcal{Z})$ associated to \mathcal{Z} is the subalgebra of $\mathcal{I}(\mathcal{Z}) \cdot \mathcal{A}'(\mathbf{Z}, \mathbf{a}) \cdot \mathcal{I}(\mathcal{Z})$ consisting of all elements α subject to the following condition. Suppose $M(a) = M(b)$, and D and D' are two diagrams, where D' is obtained from D by replacing the horizontal arc $[0, 1] \times \{a\}$ by the horizontal arc $[0, 1] \times \{b\}$. Then α contains D as a summand iff it contains D' as a summand.

We use $\mathcal{I}(\mathcal{Z})$ as the ground ring for $\mathcal{A}(\mathcal{Z})$, in the sense of Definition B.3. The condition in Definition 3.6 ensures that the canonical basis elements of $\mathcal{I}(\mathcal{Z})$ are indecomposable in $\mathcal{A}(\mathcal{Z})$.

It is straightforward to check that Definition 3.6 is equivalent to the definition of $\mathcal{A}(\mathcal{Z})$ in [Zar09].

Examples of several algebra elements are given in Figure 9. The dotted lines on the side are given to remind us of the matching in the arc diagram \mathcal{Z} . All strands are oriented left to right, so we avoid drawing them with arrows. The horizontal lines in Figure 9b are dotted, as a shorthand for the sum of two diagrams, with a single horizontal line each. For the elements in this example, we have $a_1 \cdot a_2 = a_3$, and $\partial a_1 = a_4$.

The situation for arc diagrams of β -type is completely analogous, with one important difference.

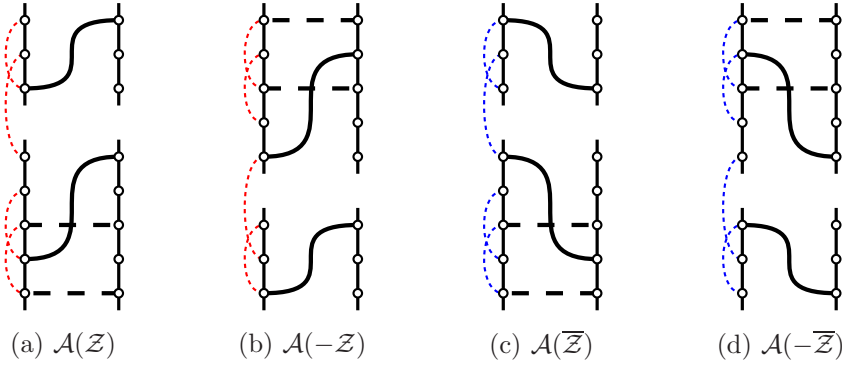


FIGURE 10. Four elements in the algebras \mathcal{Z} , $-\mathcal{Z}$, $\overline{\mathcal{Z}}$, and $-\overline{\mathcal{Z}}$, which correspond to each other.

Definition 3.7. The bordered algebra $\mathcal{A}(\mathcal{Z})$ associated to a β -arc diagram \mathcal{Z} , is defined in the exact same way as in Definitions 3.6, except that moving strands are downward veering, instead of upward.

The relationship between the different types of algebras is summarized in the following proposition.

Proposition 3.8. Suppose \mathcal{Z} is an arc diagram of either α or β -type. The algebras associated to \mathcal{Z} , $-\mathcal{Z}$, $\overline{\mathcal{Z}}$, and $-\overline{\mathcal{Z}}$ are related as follows:

$$\begin{aligned}\mathcal{A}(-\mathcal{Z}) &\cong \mathcal{A}(\overline{\mathcal{Z}}) \cong \mathcal{A}(\mathcal{Z})^{\text{op}}, \\ \mathcal{A}(-\overline{\mathcal{Z}}) &\cong \mathcal{A}(\mathcal{Z}).\end{aligned}$$

Here A^{op} denotes the opposite algebra of A . That is, an algebra with the same additive structure and differential, but the order of multiplication reversed.

Proof. This is easily seen by reflecting and rotating diagrams. To get from $\mathcal{A}(\mathcal{Z})$ to $\mathcal{A}(-\mathcal{Z})$ we have to rotate all diagrams by 180 degrees. This means that multiplication switches order, so we get the opposite algebra.

To get from $\mathcal{A}(\mathcal{Z})$ to $\mathcal{A}(\overline{\mathcal{Z}})$ we have to reflect all diagrams along the vertical axis. This again means that multiplication switches order.

An example of the correspondence is shown in Figure 10. □

3.3. The bordered invariants. We will give a brief sketch of the definitions of the bordered invariants from [Zar09], which apply for the case of α -arc diagrams. Then we discuss the necessary modifications when β -arcs are involved.

For now assume $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ is an α -arc diagram.

Definition 3.9. A bordered sutured Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \mathcal{Z})$ consists of the following:

- A compact surface Σ with no closed components.

- A collection of circles α^c and a collection of arcs α^a , which are pairwise disjoint and properly embedded in Σ . We set $\alpha = \alpha^a \cup \alpha^c$.
- A collection of disjoint circles β , properly embedded in Σ .
- An embedding $G(\mathcal{Z}) \hookrightarrow \Sigma$, such that \mathbf{Z} is sent into $\partial\Sigma$, preserving orientation, while α^a is the image of the arcs e_i in $G(\mathcal{Z})$.

We require that $\pi_0(\partial\Sigma \setminus \mathbf{Z}) \rightarrow \pi_0(\Sigma \setminus (\alpha^c \cup \alpha^a))$ and $\pi_0(\partial\Sigma \setminus \mathbf{Z}) \rightarrow \pi_0(\Sigma \setminus \beta)$ be surjective.

To such a diagram we can associate a bordered sutured manifold (Y, Γ, \mathcal{Z}) as follows. We obtain Y from $\Sigma \times [0, 1]$ by gluing 2–handles to $\beta \times \{1\}$ and $\alpha^c \times \{0\}$. The dividing set is $\Gamma = (\partial\Sigma \setminus \mathbf{Z}) \times \{1/2\}$, and $F(\mathcal{Z})$ is a neighborhood of $\mathbf{Z} \times [0, 1] \cup \alpha^a \times \{0\}$.

As proved in [Zar09], for every bordered sutured manifold there is a unique Heegaard diagram, up to isotopy and some moves.

The bordered invariants are certain homotopy-equivalence classes of \mathcal{A}_∞ –modules (see Appendix B). For a given Heegaard diagram \mathcal{H} , we can form the set of *generators* $\mathcal{G}(\mathcal{H})$ consisting of collections of intersection points of $\alpha \cap \beta$.

The invariant $\widehat{BSA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}$ is a right type- A \mathcal{A}_∞ –module over $\mathcal{A}(\mathcal{Z})$, with $\mathbb{Z}/2$ –basis $\mathcal{G}(\mathcal{H})$. The ground ring $\mathcal{I}(\mathcal{Z})$ acts as follows. The only idempotent in $\mathcal{I}(\mathcal{Z})$ which acts nontrivially on $\mathbf{x} \in \mathcal{G}(\mathcal{H})$ is $\iota_{I(\mathbf{x})}$ where $I(\mathbf{x}) \subset \{1, \dots, k\}$ records the α –arcs which contain a point of \mathbf{x} .

The structure map m of $\widehat{BSA}(\mathcal{H})$ counts certain holomorphic curves in $\text{Int } \Sigma \times [0, 1] \times \mathbb{R}$, with boundary on $(\alpha \times \{1\} \times \mathbb{R}) \cup (\beta \times \{0\} \times \mathbb{R})$. Each such curve has two types of asymptotics—ends at $(\alpha \cap \beta) \times [0, 1] \times \pm\infty$, and ends at $\partial\Sigma \times \{0\} \times \{h\}$ where $h \in \mathbb{R}$ is finite. The possible ends at $\partial\Sigma$ are in 1-to-1 correspondence with elements of $\mathcal{A}(\mathcal{Z})$.

The expression $\langle m(\mathbf{x}, a_1, \dots, a_n), \mathbf{y}^\vee \rangle$ counts curves as above, which have asymptotics $\mathbf{x} \times [0, 1]$ at $-\infty$, $\mathbf{y} \times [0, 1]$ at $+\infty$, and a_1, a_2, \dots, a_n at some finite values $h_1 < h_2 < \dots < h_n$.

We write $\widehat{BSA}(\mathcal{Y})$ for the homotopy equivalence class of $\widehat{BSA}(\mathcal{H})$. (Invariance was proven in [Zar09].)

The invariant ${}^{\mathcal{A}(-\mathcal{Z})}\widehat{BSD}(\mathcal{H})$ is a left type- D \mathcal{A}_∞ –module over $\mathcal{A}(-\mathcal{Z}) = \mathcal{A}(\mathcal{Z})^{\text{op}}$, with $\mathbb{Z}/2$ –basis $\mathcal{G}(\mathcal{H})$. (See Appendix B.2 for type- D modules, and the meaning of upper and lower indices). The ground ring $\mathcal{I}(-\mathcal{Z})$ acts as follows. The only idempotent in $\mathcal{I}(-\mathcal{Z})$ which acts nontrivially on $\mathbf{x} \in \mathcal{G}(\mathcal{H})$ is $\iota_{I^c(\mathbf{x})}$ where $I^c(\mathbf{x}) \subset \{1, \dots, k\}$ records the α –arcs which *do not* contain a point of \mathbf{x} .

The structure map δ of $\widehat{BSD}(\mathcal{H})$ counts a subset of the same holomorphic curves as for $\widehat{BSA}(\mathcal{H})$. Their interpretation is somewhat different, though. Equivalently, ${}^{\mathcal{A}(\mathcal{Z})^{\text{op}}}\widehat{BSD}(\mathcal{H}) = \widehat{BSA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z}), \mathcal{A}(\mathcal{Z})^{\text{op}}}\mathbb{I}$, where \mathbb{I} is a certain bimodule defined in [LOT10a].

Again, we write $\widehat{BSD}(\mathcal{Y})$ for the homotopy equivalence class of $\widehat{BSD}(\mathcal{H})$. (Invariance was proven in [Zar09].)

We can also construct invariants ${}_{\mathcal{A}(\mathcal{Z})^{\text{op}}} \widehat{BSA}(\mathcal{Y})$ and $\widehat{BSD}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z})}$ purely algebraically from the usual \widehat{BSA} and \widehat{BSD} . Indeed, as discussed in Appendix B.6, any right A -module is a left- A^{op} module and vice versa.

If \mathcal{Y} is bordered by $\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$, we can similarly define several bimodules invariants for \mathcal{Y} :

$$\begin{array}{cc} {}_{\mathcal{A}(\mathcal{Z}_1)^{\text{op}}} \widehat{BSAA}(\mathcal{Y})_{\mathcal{A}(\mathcal{Z}_2)} & {}^{\mathcal{A}(\mathcal{Z}_1)^{\text{op}}} \widehat{BSDA}(\mathcal{Y})_{\mathcal{A}(\mathcal{Z}_2)} \\ {}_{\mathcal{A}(\mathcal{Z}_1)^{\text{op}}} \widehat{BSAD}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z}_2)} & {}^{\mathcal{A}(\mathcal{Z}_1)^{\text{op}}} \widehat{BSD\bar{D}}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z}_2)} \end{array}$$

For the invariants of β -diagrams little changes. Suppose \mathcal{Z} is a β -type arc diagram. Heegaard diagrams will now involve β -arcs as the images of $e_i \subset G(\mathcal{Z})$, instead of α -arcs. We still count holomorphic curves in $\text{Int } \Sigma \times [0, 1] \times \mathbb{R}$. However, since there are β -curves hitting $\partial\Sigma$ instead of α , the asymptotic ends at $\partial\Sigma \times \{1\} \times \{h\}$ are replaced by ends at $\partial\Sigma \times \{0\} \times \{h\}$, which again correspond to elements of $\mathcal{A}(\mathcal{Z})$. The rest of the definition is essentially unchanged.

The last case is when \mathcal{Y} is bordered by $\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$, where \mathcal{Z}_1 is a diagram of α -type and \mathcal{Z}_2 is of β -type. We can extend the definition of $\widehat{BSAA}(\mathcal{Y})$ as before. There are now four types of asymptotic ends:

- The ones at $\pm\infty$ which correspond to generators $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\mathcal{H})$.
- $\partial\Sigma \times \{1\} \times \{h\}$ (or α -ends) which correspond to $\mathcal{A}(\mathcal{Z}_1)$.
- $\partial\Sigma \times \{0\} \times \{h\}$ (or β -ends) which correspond to $\mathcal{A}(\mathcal{Z}_2)$.

Each holomorphic curve will have some number $k \geq 0$ of α -ends, and some number $l \geq 0$ of β -ends. Such a curve contributes to the structure map $m_{k|1|l}$ which takes k elements of $\mathcal{A}(\mathcal{Z}_1)$ and l elements of $\mathcal{A}(\mathcal{Z}_2)$.

To summarize we have the following theorem.

Theorem 3.10. *Let \mathcal{Y} be a bordered sutured manifold, bordered by $-\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$, where \mathcal{Z}_1 and \mathcal{Z}_2 can be any combination of α and β types. Then there are bimodules, well defined up to homotopy equivalence:*

$$\begin{array}{cc} {}_{\mathcal{A}(\mathcal{Z}_1)} \widehat{BSAA}(\mathcal{Y})_{\mathcal{A}(\mathcal{Z}_2)} & {}^{\mathcal{A}(\mathcal{Z}_1)} \widehat{BSDA}(\mathcal{Y})_{\mathcal{A}(\mathcal{Z}_2)} \\ {}_{\mathcal{A}(\mathcal{Z}_1)} \widehat{BSAD}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z}_2)} & {}^{\mathcal{A}(\mathcal{Z}_1)} \widehat{BSD\bar{D}}(\mathcal{Y})^{\mathcal{A}(\mathcal{Z}_2)} \end{array}$$

If \mathcal{Y}_1 and \mathcal{Y}_2 are two such manifolds, bordered by $-\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$ and $-\mathcal{F}(\mathcal{Z}_2) \sqcup \mathcal{F}(\mathcal{Z}_3)$, respectively, then there are homotopy equivalences

$$\begin{aligned} \widehat{BSAA}(\mathcal{Y}_1 \cup \mathcal{Y}_2) &\simeq \widehat{BSAA}(\mathcal{Y}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{BSDA}(\mathcal{Y}_2), \\ \widehat{BSDA}(\mathcal{Y}_1 \cup \mathcal{Y}_2) &\simeq \widehat{BSD\bar{D}}(\mathcal{Y}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{BSAA}(\mathcal{Y}_2), \end{aligned}$$

etc. Any combination of bimodules for \mathcal{Y}_1 and \mathcal{Y}_2 can be used, where one is type- A for $\mathcal{A}(\mathcal{Z}_2)$, and the other is type- D for $\mathcal{A}(\mathcal{Z}_2)$.

The latter statement is referred to as the *pairing theorem*. The proof of Proposition 3.10 is a straightforward adaptation of the corresponding proofs when dealing with only type- α diagrams. An analogous construction involving both α and β arcs in the purely bordered setting is given in [LOT10b].

3.4. Mirrors and twisting slices. In this section we give two computations of bordered invariants. One of them relates the invariants for a bordered sutured manifold \mathcal{W} and its mirror $-\mathcal{W}$. The other gives the invariants for a positive and negative twisting slice.

Recall that if $\mathcal{W} = (W, \Gamma, \mathcal{F}(\mathcal{Z}))$, its mirror is $-\mathcal{W} = (-W, \Gamma, \overline{\mathcal{F}(\mathcal{Z})}) = (-W, \Gamma, \mathcal{F}(\overline{\mathcal{Z}}))$.

Proposition 3.11. *Let \mathcal{W} and $-\mathcal{W}$ be as above. Let $M_{\mathcal{A}(\mathcal{Z})}$ be a representative for the homotopy equivalence class $\widehat{BSA}(\mathcal{W})_{\mathcal{A}(\mathcal{Z})}$. Then its dual $_{\mathcal{A}(\mathcal{Z})}M^\vee$ is a representative for $_{\mathcal{A}(\mathcal{Z})}\widehat{BSA}(-\mathcal{W})$. Similarly, there are homotopy equivalences*

$$\begin{aligned} \left(\widehat{BSD}(\mathcal{W})^{\mathcal{A}(\mathcal{Z})} \right)^\vee &\simeq {}^{\mathcal{A}(\mathcal{Z})}\widehat{BSD}(-\mathcal{W}), \\ \left({}_{\mathcal{A}(\mathcal{Z})^{\text{op}}}\widehat{BSA}(\mathcal{W}) \right)^\vee &\simeq \widehat{BSA}(-\mathcal{W})_{\mathcal{A}(\mathcal{Z})^{\text{op}}}, \\ \left({}^{\mathcal{A}(\mathcal{Z})^{\text{op}}}\widehat{BSD}(\mathcal{W}) \right)^\vee &\simeq \widehat{BSD}(-\mathcal{W})^{\mathcal{A}(\mathcal{Z})^{\text{op}}}. \end{aligned}$$

A similar statement holds for bimodules—if \mathcal{W} is bordered by $\mathcal{F}(\mathcal{Z}_1) \sqcup \mathcal{F}(\mathcal{Z}_2)$, then the corresponding bimodule invariants of \mathcal{W} and $-\mathcal{W}$ are duals of each other.

Proof. We prove one case. All others follow by analogy. Let $\mathcal{H} = (\Sigma, \alpha, \beta, \mathcal{Z})$ be a Heegaard diagram for \mathcal{W} . Let $\mathcal{H}' = (\Sigma, \beta, \alpha, \overline{\mathcal{Z}})$ be the diagram obtained by switching all α and β curves. (Note that if \mathcal{Z} was an α -type diagram, this turns it into the β -type diagram $\overline{\mathcal{Z}}$, and vice versa.)

The bordered sutured manifold described by \mathcal{H}' is precisely $-\mathcal{W}$. Indeed, it is obtained from the same manifold $\Sigma \times [0, 1]$ by attaching all 2-handles on the opposite side, and taking the sutured surface \mathcal{F} also on the opposite side. This is equivalent to reversing the orientation of W , while keeping the orientations of $\Gamma \subset \partial\Sigma$ and $\mathbf{Z} \subset \partial\Sigma$ the same. (Compare to [HKM07], where the EH -invariant for contact structures on (Y, Γ) is defined in $SFH(-Y, +\Gamma)$.)

The generators $\mathcal{G}(\mathcal{H})$ and $\mathcal{G}(\mathcal{H}')$ of the two diagrams are the same. There is also a 1-to-1 correspondence between the holomorphic curves u in the definition of $\widehat{BSA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}$ and the curves u' in the definition of $\widehat{BSA}(\mathcal{H}')_{\mathcal{A}(\overline{\mathcal{Z}})}$. This is given by reflecting both the $[0, 1]$ -factor and the \mathbb{R} -factor in the domain $\text{Int } \Sigma \times [0, 1] \times \mathbb{R}$. The $\pm\infty$ asymptotic ends are reversed. The α -ends of u are sent to the β -ends of u' , and vice versa, while their heights h on the \mathbb{R} -scale are reversed. When turning α -ends to β -ends, the corresponding elements of $\mathcal{A}(\mathcal{Z})$ are reflected (as in the correspondence $\mathcal{A}(\overline{\mathcal{Z}}) \cong \mathcal{A}(\mathcal{Z})^{\text{op}}$ from Proposition 3.8).

This implies the following relation between the structure maps m of $\widehat{BSA}(\mathcal{H})$ and m' of $\widehat{BSA}(\mathcal{H}')$:

$$\langle m(\mathbf{x}, a_1, \dots, a_n), \mathbf{y}^\vee \rangle = \langle m'(\mathbf{y}', a_n^{\text{op}}, \dots, a_1^{\text{op}}), \mathbf{x}^\vee \rangle.$$

Turning $\widehat{BSA}(\mathcal{H}')$ into a left module over $(\mathcal{A}(\mathcal{Z})^{\text{op}})^{\text{op}} = \mathcal{A}(\mathcal{Z})$, we get the relation

$$\langle m(\mathbf{x}, a_1, \dots, a_n), \mathbf{y}^\vee \rangle = \langle m'(a_1, \dots, a_n, \mathbf{y}'), \mathbf{x}'^\vee \rangle.$$

This is precisely the statement that $\widehat{BSA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z})}$ and $_{\mathcal{A}(\mathcal{Z})}\widehat{BSA}(\mathcal{H}')$ are duals, with $\mathcal{G}(\mathcal{H})$ and $\mathcal{G}(\mathcal{H}')$ as dual bases. \square

A similar statement for purely bordered invariants is proven in [LOT10b].

Proposition 3.12. *Let \mathcal{Z} be any arc diagram, and let $A = \mathcal{A}(\mathcal{Z})$. The twisting slices $\mathcal{TW}_{\mathcal{F}(\mathcal{Z}), \pm}$ are bordered by $-\mathcal{F}(\mathcal{Z}) \sqcup -\overline{\mathcal{F}(\mathcal{Z})}$. They have bimodule invariants*

$$_A \widehat{BSAA}(\mathcal{TW}_{\mathcal{F}(\mathcal{Z}), -})_A \simeq {}_A A_A, \quad {}_A \widehat{BSAA}(\mathcal{TW}_{\mathcal{F}(\mathcal{Z}), +})_A \simeq {}_A A^\vee_A.$$

Proof. Since $\mathcal{TW}_{\mathcal{F}(\mathcal{Z}), \pm}$ are mirrors of each other, by Proposition 3.11, it is enough to prove the first equivalence. The key ingredient is a very convenient nice diagram \mathcal{H} for $\mathcal{TW}_{\mathcal{F}(\mathcal{Z}), -}$. This diagram was discovered by the author, and independently by Auroux in [Aur], where it appears in a rather different setting.

Recall from [Zar09] that a nice diagram is a diagram, $(\Sigma, \alpha, \beta, \mathcal{Z})$ where each region of $\Sigma \setminus (\alpha \cup \beta)$ is either a boundary region, a rectangle, or a bigon. The definition trivially extends to the current more general setting. Nice diagrams can still be used to combinatorially compute bordered sutured invariants.

The diagram is obtained as follows. For concreteness assume that \mathcal{Z} is of α -type. To construct the Heegaard surface Σ , start with several squares $[0, 1] \times [0, 1]$, one for each component $Z \in \mathbf{Z}$. There are three identifications of \mathbf{Z} with sides of the squares:

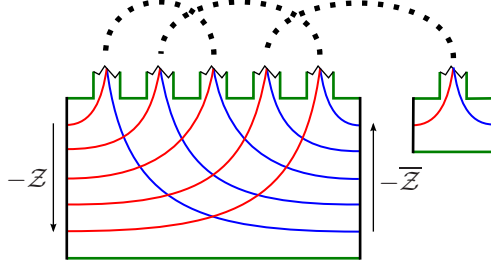
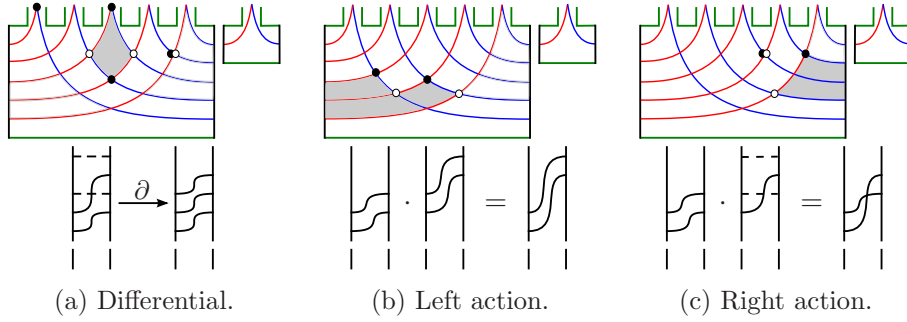
- φ sending \mathbf{Z} to the “left sides” $\{0\} \times [0, 1]$, oriented from 0 to 1.
- φ' sending \mathbf{Z} to the “right sides” $\{1\} \times [0, 1]$, oriented from 1 to 0.
- ψ sending \mathbf{Z} to the “top sides” $[0, 1] \times \{1\}$, oriented from 1 to 0.

For each matched pair $\{a, b\} = M^{-1}(i) \subset \mathbf{a} \subset \mathbf{Z}$, attach a 1-handle at $\psi(\{a, b\})$. Add an α -arc α_i^a from $\varphi(a)$ to $\varphi(b)$, and a β -arc β_i^a from $\varphi'(a)$ to $\varphi'(b)$, both running through the handle corresponding to a, b . To see that this gives the correct manifold, notice that there are no α or β -circles, so the manifold is topologically $\Sigma \times [0, 1]$. The pattern of attachment of the 1-handles shows that $\Sigma = F(\mathcal{Z})$. It is easy to check that Γ and the arcs are in the correct positions.

This construction is demonstrated in Figure 11. The figure corresponds to the arc diagram \mathcal{Z} from Figure 7c.

Calculations with the same diagram in [Aur] and [LOT10b] show that the bimodule $\widehat{BSAA}(\mathcal{H})$ is indeed the algebra A as a bimodule over itself. While the statements in those cases are not about bordered sutured Floer homology, the argument is purely combinatorial and carries over completely.

We give a brief summary of this argument. Intersection points in $\alpha \cap \beta$ are of two types:

FIGURE 11. Heegaard diagram for a negative twisted slice $\mathcal{TW}_{\mathcal{F},-}$.FIGURE 12. Examples of domains counted in the diagram for $\mathcal{TW}_{\mathcal{F},-}$. In each case the domain goes from the black dots to the white dots. Below them we show the corresponding operations on the algebra.

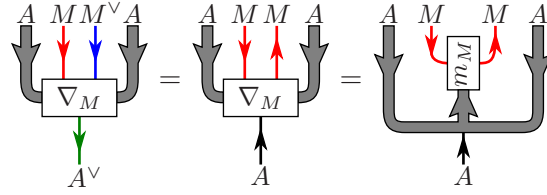
- $x_i \in \alpha_i^a \cap \beta_i^a$ inside the 1-handle corresponding to $M^{-1}(i)$, for $i \in \{1, \dots, k\}$. The point x_i corresponds to the two horizontal strands $[0, 1] \times M^{-1}(i)$ in $\mathcal{A}(\mathcal{Z})$.
- $y_{ab} \in \alpha_{M(a)}^a \cap \beta_{M(b)}^a$, inside the square regions of \mathcal{H} . The point y_{ab} corresponds to a strand $(0, a) \rightarrow (1, b)$ (or $a \rightarrow b$ for short) in $\mathcal{A}(\mathcal{Z})$.

The allowed combinations of intersection points correspond to the allowed diagrams in $\mathcal{A}(\mathcal{Z})$, so $\widehat{BSA}(\mathcal{H}) \cong \mathcal{A}(\mathcal{Z})$ as a $\mathbb{Z}/2$ -vector space.

Since \mathcal{H} is a nice diagram the differential counts embedded rectangles in \mathcal{H} , with sides on α and β . The rectangle with corners $(y_{ad}, y_{bc}, y_{ac}, y_{ad})$ corresponds to resolving the crossing between the strands $a \rightarrow d$ and $b \rightarrow c$ (getting $a \rightarrow c$ and $b \rightarrow d$).

The left action $m_{1|1|0}$ of A counts rectangles hitting the $-\mathcal{Z}$ -part of the boundary. The rectangle with corners $(\varphi(a), y_{ac}, y_{bc}, \varphi(b))$ corresponds to concatenating the strands $a \rightarrow b$ and $b \rightarrow c$ (getting $a \rightarrow c$). The right action is similar, with rectangles hitting the $-\overline{\mathcal{Z}}$ -part of the boundary.

Some examples of domains in \mathcal{H} contributing to $m_{0|1|0}$, $m_{1|1|0}$, and $m_{0|1|1}$ are shown in Figure 12. They are for the diagram \mathcal{H} from Figure 11. \square

FIGURE 13. Definition of the join map ∇ .

4. THE JOIN MAP

In this section we will define the join and gluing maps, and prove some basic properties. Recall that the gluing operation is defined as a special case of the join operation. The gluing map is similarly a special case of the join map. Thus for the most part we will only talk about the general case, i.e. the join map.

4.1. The algebraic map. We will first define an abstract algebraic map, on the level of \mathcal{A}_∞ -modules.

Let A be a differential graded algebra, and ${}_A M$ be a left \mathcal{A}_∞ -module over it. As discussed in Appendix B.6, the dual M^\vee_A is a right \mathcal{A}_∞ -module over A . Thus ${}_A(M \otimes M^\vee)_A$ is an \mathcal{A}_∞ -bimodule. On the other hand, since A is a bimodule over itself, so is its dual ${}_A A^\vee_A$. We define a map $M \otimes M^\vee \rightarrow A^\vee$ which is an \mathcal{A}_∞ -analog of the natural pairing of a module and its dual.

Definition 4.1. *The algebraic join map $\nabla_M: {}_A(M \otimes M^\vee)_A \rightarrow {}_A A^\vee_A$ —or just ∇ when unambiguous—is an \mathcal{A}_∞ -bimodule morphism, defined as follows. It is the unique morphism satisfying*

$$(1) \quad \langle \nabla_{i|1|j}(a_1, \dots, a_i, p, q^\vee, a'_1, \dots, a'_j), a'' \rangle \\ = \langle m_{i+j+1|1}(a'_1, \dots, a'_j, a'', a_1, \dots, a_i, p), q^\vee \rangle,$$

for any $i, j \geq 0$, $p \in M$, $q^\vee \in M^\vee$, and $a_* \in A$.

Eq. (1) is best represented diagrammatically, as in Figure 13. Note that ∇_M is a bounded morphism if and only if M is a bounded module.

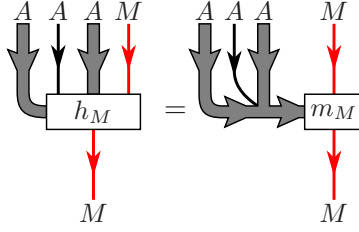
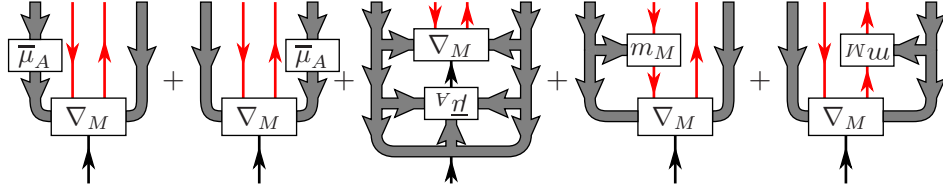
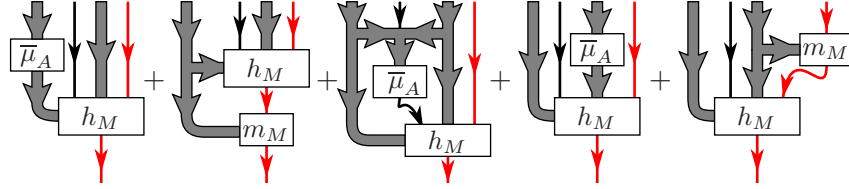
As discussed in Appendix B.4, morphisms of \mathcal{A}_∞ -modules form chain complexes, where cycles are *homomorphisms*. Only homomorphisms descend to maps on homology.

Proposition 4.2. *For any ${}_A M$, the join map ∇_M is a homomorphism.*

Proof. It is a straightforward but tedious computation to see that $\partial \nabla_M = 0$ is equivalent to the structure equation for m_M .

A more enlightening way to see this is to notice that by turning the diagram in Figure 13 partly sideways, we get a diagram for the homotopy equivalence $h_M: A \tilde{\otimes} M \rightarrow M$, shown in Figure 14. Taking the differential $\partial \nabla_M$ and turning the resulting diagrams sideways, we get precisely ∂h_M . We know that h_M is a homomorphism and, so $\partial h_M = 0$.

The equivalences are presented in Figure 15. □

FIGURE 14. The homotopy equivalence $h_M: A \otimes M \rightarrow M$.(a) The differential $\partial \nabla_M$ which needs to vanish to show that ∇_M is an \mathcal{A}_∞ -bimodule homomorphism.(b) The differential ∂h_M of the homotopy equivalence h_M .FIGURE 15. Proof that ∇ is a homomorphism, by rotating diagrams.

We will prove two naturality statements about ∇ that together imply that ∇ descends to a well defined map on the derived category. The first shows that ∇ is natural with respect to isomorphisms in the derived category of the DG-algebra A , i.e. homotopy equivalences of modules. The second shows that ∇ is natural with respect to equivalences of derived categories. (Recall from [Zar09] that different algebras corresponding to the same sutured surface are derived-equivalent.)

Proposition 4.3. *Suppose ${}_A M$ and ${}_A N$ are two \mathcal{A}_∞ -modules over A , such that there are inverse homotopy equivalences $\varphi: M \rightarrow N$ and $\psi: N \rightarrow M$. Then there is an \mathcal{A}_∞ -homotopy equivalence of A, A -bimodules*

$$\varphi \otimes \psi^\vee: M \otimes M^\vee \rightarrow N \otimes N^\vee,$$

and the following diagram commutes up to \mathcal{A}_∞ -homotopy:

$$\begin{array}{ccc} M \otimes M^\vee & & \\ \varphi \otimes \psi^\vee \downarrow & \searrow \nabla_M & \\ N \otimes N^\vee & \xrightarrow{\nabla_N} & A^\vee. \end{array}$$

Proposition 4.4. *Suppose A and B are differential graded algebras, and ${}_B X^A$ and ${}_A Y^B$ are two type-DA bimodules, which are quasi-inverses. That is, there are \mathcal{A}_∞ -homotopy equivalences*

$${}_A(Y \boxtimes X)^A \simeq {}_A \mathbb{I}^A, \quad {}_B(X \boxtimes Y)^B \simeq {}_B \mathbb{I}^B.$$

Moreover, suppose $H_(B^\vee)$ and $H_*(X \boxtimes A^\vee \boxtimes X^\vee)$ have the same rank (over $\mathbb{Z}/2$).*

Then there is a B, B -bimodule homotopy equivalence

$$\varphi_X: X \boxtimes A^\vee \boxtimes X^\vee \rightarrow B^\vee.$$

Moreover, for any \mathcal{A}_∞ -module ${}_A M$, such that $X \boxtimes M$ is well defined, the following diagram commutes up to \mathcal{A}_∞ -homotopy:

$$\begin{array}{ccc} X \boxtimes M \otimes M^\vee \boxtimes X^\vee & & \\ \text{id}_X \boxtimes \nabla_M \boxtimes \text{id}_{X^\vee} \downarrow & \searrow \nabla_{X \boxtimes M} & \\ X \boxtimes A^\vee \boxtimes X^\vee & \xrightarrow{\varphi_X} & B^\vee. \end{array}$$

Notice the condition that $X \boxtimes M$ be well defined. This can be satisfied for example if M is a bounded module, or if X is relatively bounded in A with respect to B . Before proving Propositions 4.3 and 4.4 in Section 4.3, we will use them to define the join Ψ .

4.2. The geometric map. Suppose that \mathcal{Y}_1 and \mathcal{Y}_2 are two sutured manifolds, and $\mathcal{W} = (W, \Gamma, -\mathcal{F})$ is a partially sutured manifold, with embeddings $\mathcal{W} \hookrightarrow \mathcal{Y}_1$ and $-\mathcal{W} \hookrightarrow \mathcal{Y}_2$. Let \mathcal{Z} be any arc diagram parametrizing the surface \mathcal{F} . Recall that $-\mathcal{W} = (-W, \Gamma, -\overline{\mathcal{F}})$. Also recall the twisting slice $\mathcal{TW}_{\mathcal{F},+}$, from Definition 2.8. The join $\mathcal{Y}_1 \mathbin{\mathbb{U}}_{\mathcal{W}} \mathcal{Y}_2$ of \mathcal{Y}_1 and \mathcal{Y}_2 along \mathcal{W} was defined as

$$\mathcal{Y}_1 \mathbin{\mathbb{U}}_{\mathcal{W}} \mathcal{Y}_2 = (\mathcal{Y}_1 \setminus \mathcal{W}) \cup_{\mathcal{F}} \mathcal{TW}_{\mathcal{F},+} \cup_{-\overline{\mathcal{F}}} (\mathcal{Y}_2 \setminus -\mathcal{W}).$$

Let $A = \mathcal{A}(\mathcal{Z})$ be the algebra associated to \mathcal{Z} . Let ${}_A M$, U^A , and ${}^A V$ be representatives for the bordered sutured modules ${}_A \widehat{BSA}(\mathcal{W})$, $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W})^A$, and ${}^A \widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W})$, respectively such that $U \boxtimes M$ and $M^\vee \boxtimes V$ are well-defined. (Recall that the modules are only defined up to homotopy equivalence, and that the \boxtimes product is only defined under some boundedness conditions.) We proved in Proposition 3.11 that M^\vee_A is a representative for $\widehat{BSA}(-\mathcal{W})_A$, and in Proposition 3.12 that ${}_A A^\vee_A$ is a representative for $\widehat{BSAA}(\mathcal{TW}_{\mathcal{F},+})$.

From the Künneth formula for SFH of a disjoint union, and from Theorem 3.10, we have the following homotopy equivalences of chain complexes.

$$\begin{aligned} SFC(\mathcal{Y}_1 \cup \mathcal{Y}_2) &\cong SFC(\mathcal{Y}_1) \otimes SFC(\mathcal{Y}_2) \\ &\simeq \left(\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W}) \boxtimes_A \widehat{BSA}(\mathcal{W}) \right) \otimes \left(\widehat{BSA}(-\mathcal{W}) \boxtimes_A \widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W}) \right) \\ &\simeq U^A \boxtimes {}_A (M \otimes M^\vee)_A \boxtimes {}^A V. \end{aligned}$$

$$\begin{aligned}
SFC(\mathcal{Y}_1 \uplus_{\mathcal{W}} \mathcal{Y}_2) \\
\simeq \widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W}) \boxtimes_A \widehat{BSAA}(\mathcal{TW}_{\mathcal{F},+}) \boxtimes_A \widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W}) \\
\simeq U^A \boxtimes_A A^\vee_A \boxtimes A^V.
\end{aligned}$$

Definition 4.5. Let \mathcal{Y}_1 , \mathcal{Y}_2 and \mathcal{W} be as described above. Define the geometric join map

$$\Psi_M: SFC(\mathcal{Y}_1) \otimes SFC(\mathcal{Y}_2) \rightarrow SFC(\mathcal{Y}_1 \uplus_{\mathcal{W}} \mathcal{Y}_2)$$

by the formula

$$(2) \quad \Psi_M = \text{id}_U \boxtimes \nabla_M \boxtimes \text{id}_V: U \boxtimes M \otimes M^\vee \boxtimes V \rightarrow U \boxtimes A^\vee \boxtimes V.$$

Note that such an induced map is not generally well defined (it might involve an infinite sum). In this case, however, we have made some boundedness assumptions. Since $U \boxtimes M$ and $M^\vee \boxtimes V$ are defined, either M must be bounded, or both of U and V must be bounded. In the former case, ∇_M is also bounded. Either of these situations guarantees that the sum defining Ψ_M is finite.

Theorem 4.6. The map Ψ_M from Definition 4.5 is, up to homotopy, independent on the choice of parametrization \mathcal{Z} , and on the choices of representatives M , U , and V .

Proof. First, we will give a more precise version of the statement. Let \mathcal{Z}' be any other parametrization of \mathcal{F} , with $B = \mathcal{A}(-\mathcal{Z}')$, and let ${}_B M'$, U'^B and ${}_B V'$, be representatives for the respective bordered sutured modules. Then there are homotopy equivalences φ and ψ making the following diagram commute up to \mathcal{A}_∞ -homotopy:

$$\begin{array}{ccc}
U \boxtimes M \otimes M^\vee \boxtimes V & \xrightarrow{\varphi} & U' \boxtimes M' \otimes M'^\vee \boxtimes V' \\
\Psi_M \downarrow & & \downarrow \Psi_{M'} \\
U \boxtimes A^\vee \boxtimes V & \xrightarrow{\psi} & U' \boxtimes B^\vee \boxtimes V'.
\end{array}$$

The proof can be broken up into several steps. The first step is independence from the choice of U and V , given a fixed choice for A and M . This follows directly from the fact $\text{id} \boxtimes \cdot$ and $\cdot \boxtimes \text{id}$ are DG-functors.

The second step is to show independence from the choice of M , for fixed A , U , and V . This follows from Proposition 4.3. Indeed, suppose $\varphi: M \rightarrow M'$ is a homotopy equivalence with homotopy inverse $\psi: M' \rightarrow M$. Then $\psi^\vee: M^\vee \rightarrow M'^\vee$ is also a homotopy equivalence inducing the homotopy equivalence

$$\text{id}_U \boxtimes \varphi \otimes \psi^\vee \boxtimes \text{id}_V: U \boxtimes M \otimes M^\vee \boxtimes V \rightarrow U \boxtimes M' \otimes M'^\vee \boxtimes V.$$

By Proposition 4.3, $\nabla_M \simeq \nabla_{M'} \circ (\varphi \otimes \psi^\vee)$, which implies

$$\text{id}_U \boxtimes \nabla_M \boxtimes \text{id}_V \simeq (\text{id}_U \boxtimes \nabla_{M'} \boxtimes \text{id}_V) \circ (\text{id}_U \boxtimes \varphi \otimes \psi^\vee \boxtimes \text{id}_V).$$

The final step is to show independence from the choice of algebra A . We will cut \mathcal{Y}_1 and \mathcal{Y}_2 into several pieces, so we can evaluate the two different versions of Ψ from the same geometric picture.

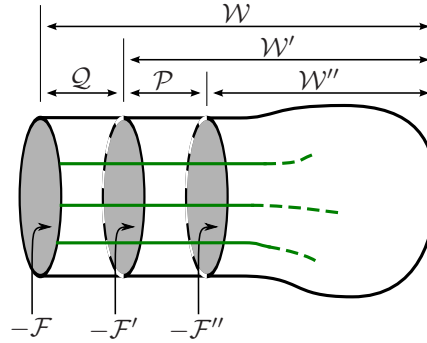


FIGURE 16. The various pieces produced by slicing \mathcal{W} at two surfaces parallel to \mathcal{F} .

Let $-\mathcal{F}'$ and $-\mathcal{F}''$ be two parallel copies of $-\mathcal{F}$ in \mathcal{W} , which cut out $\mathcal{W}' = (W', \Gamma', -\mathcal{F}')$ and $\mathcal{W}'' = (W'', \Gamma'', -\mathcal{F}'')$, where $\mathcal{W}'' \subset \mathcal{W}' \subset \mathcal{W}$. Let $\mathcal{P} = \mathcal{W}' \setminus \mathcal{W}''$ and $\mathcal{Q} = \mathcal{W} \setminus \mathcal{W}'$ (see Figure 16). Both \mathcal{P} and \mathcal{Q} are topologically $F \times [0, 1]$.

Parametrize \mathcal{F} and \mathcal{F}'' by \mathcal{Z} , and \mathcal{F}' by \mathcal{Z}' , where $\mathcal{A}(\mathcal{Z}) = A$, and $\mathcal{A}(\mathcal{Z}') = B$. Let ${}_B X^A$ and ${}_A Y^B$ be representatives for ${}_B \widehat{BSAD}(\mathcal{P})^A$ and ${}_A \widehat{BSAD}(\mathcal{Q})^B$, respectively. Note that $\mathcal{Q} \cup_{\mathcal{F}'} \mathcal{P}$ is a product bordered sutured manifold, and thus has trivial invariant ${}_A \widehat{BSAD}(\mathcal{Q} \cup \mathcal{P})^A \simeq {}_A \mathbb{I}^A$. By the pairing theorem, this implies $Y \boxtimes X \simeq {}_A \mathbb{I}^A$. Similarly, by stacking \mathcal{P} and \mathcal{Q} in the opposite order we get $X \boxtimes Y \simeq {}_B \mathbb{I}^B$.

There are embeddings $\mathcal{W}', \mathcal{W}'' \hookrightarrow \mathcal{Y}_1$ and $-\mathcal{W}', -\mathcal{W}'' \hookrightarrow \mathcal{Y}_2$ and two distinct ways to cut and glue them together, getting $\mathcal{Y}_1 \uplus_{\mathcal{W}'} \mathcal{Y}_2 \cong \mathcal{Y}_1 \uplus_{\mathcal{W}''} \mathcal{Y}_2$. This is illustrated schematically in Figure 17.

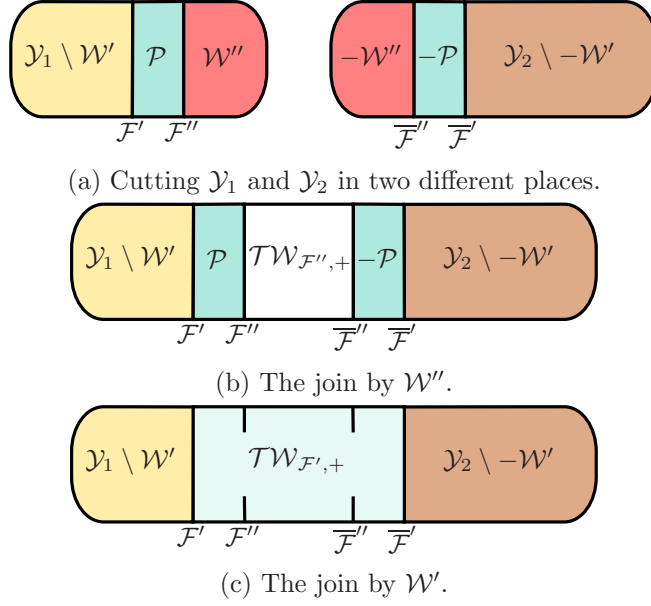
Let ${}_A M$ be a representative for ${}_A \widehat{BSA}(\mathcal{W}'')$. By the pairing theorem, ${}_B(X \boxtimes M)$ is a representative for ${}_B \widehat{BSA}(\mathcal{W}')$. Notice that $\mathcal{TW}_{\mathcal{F}',+} \cong \mathcal{P} \cup \mathcal{TW}_{\mathcal{F}'',+} \cup -\mathcal{P}$ and ${}_B B^\vee_B$ and ${}_B(X \boxtimes A^\vee \boxtimes X^\vee)_B$ are both representatives for its \widehat{BSAA} invariant. In particular, they have the same homology. Finally, let U^B and ${}^B V$ be representatives for $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W}')^B$ and ${}^B \widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W}')$, respectively.

The two join maps Ψ_M and $\Psi_{X \boxtimes M}$ are described by the following equations.

$$\begin{aligned} \Psi_M &= \text{id}_{U \boxtimes X} \boxtimes \nabla_M \boxtimes \text{id}_{X^\vee \boxtimes V} : \\ & (U \boxtimes X) \boxtimes M \otimes M^\vee \boxtimes (X^\vee \boxtimes V) \rightarrow (U \boxtimes X) \boxtimes A^\vee \boxtimes (X^\vee \boxtimes V), \end{aligned}$$

$$\begin{aligned} \Psi_{X \boxtimes M} &= \text{id}_U \boxtimes \nabla_{X \boxtimes M} \boxtimes \text{id}_V : \\ & U \boxtimes (X \boxtimes M) \otimes (M^\vee \boxtimes X^\vee) \boxtimes V \rightarrow U \boxtimes B^\vee \boxtimes V. \end{aligned}$$

We can apply Proposition 4.4. The boundedness condition can be satisfied by requiring that X and Y are bounded modules. There is a homotopy

FIGURE 17. Two ways of cutting and pasting to get the join of \mathcal{Y}_1 and \mathcal{Y}_2 .

equivalence $\varphi_X: X \boxtimes A^\vee \boxtimes X^\vee \rightarrow B$, and a homotopy $\nabla_{X \boxtimes M} \sim \varphi_X \circ (\text{id}_X \boxtimes \nabla_M \boxtimes \text{id}_{X^\vee})$. These induce a homotopy

$$\begin{aligned} (\text{id}_U \boxtimes \varphi_X \boxtimes \text{id}_V) \circ \Psi_M &= \text{id}_U \boxtimes (\varphi_X \circ (\text{id}_X \boxtimes \nabla_M \boxtimes \text{id}_{X^\vee})) \boxtimes \text{id}_V \\ &\sim \text{id}_U \boxtimes \nabla_{X \boxtimes M} \boxtimes \text{id}_V = \Psi_{X \boxtimes M}. \end{aligned}$$

This finishes the last step. Combining all three gives complete invariance. Thus we can refer to $\Psi_{\mathcal{W}}$ from now on. \square

4.3. Proof of algebraic invariance. In this section we prove Propositions 4.3 and 4.4.

Proof of Proposition 4.3. The proof will be mostly diagrammatic. There are two modules ${}_A M$ and ${}_A N$, and two inverse homotopy equivalences, $\varphi: M \rightarrow N$ and $\psi: N \rightarrow M$. The dualizing functor ${}_A \text{Mod} \rightarrow \text{Mod}_A$ is a DG-functor. Thus it is easy to see that

$$\varphi \otimes \psi^\vee = (\varphi \otimes \text{id}_{N^\vee}) \circ (\text{id}_M \otimes \psi^\vee)$$

is also a homotopy equivalence. Let $H: M \rightarrow M$ be the homotopy between id_M and $\psi \circ \varphi$.

We have to show that the homomorphism

$$(3) \quad \nabla_M + \nabla_N \circ (\varphi \otimes \psi^\vee)$$

is null-homotopic (see Figure 18a). Again, it helps if we turn the diagram sideways, where bar resolutions come into play. Let $h_M: A \tilde{\otimes} M \rightarrow M$ and $h_N: A \tilde{\otimes} N \rightarrow N$ be the natural homotopy equivalences.

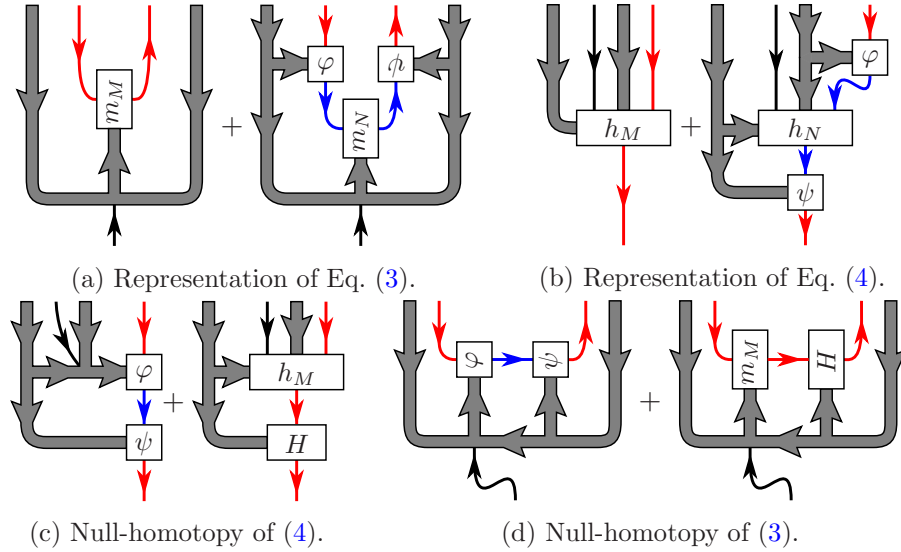


FIGURE 18. Diagrams from the proof of Proposition 4.3.

Turning the first term in Eq. (3) sideways, we get h_M . Turning the second term sideways we get $\psi \circ h_N \circ (\text{id}_A \tilde{\otimes} \varphi)$. Thus we need to show that

$$(4) \quad h_M + \psi \circ h_N \circ (\text{id}_A \tilde{\otimes} \varphi)$$

is null-homotopic (see Figure 18b).

There is a canonical homotopy $h_\varphi: A \tilde{\otimes} M \rightarrow N$ between $\varphi \circ h_M$ and $h_N \circ (\text{id}_A \tilde{\otimes} \varphi)$, given by

$$h_\varphi(a_1, \dots, a_i, (a', a''_1, \dots, a''_j, m)) = \varphi(a_1, \dots, a_i, a', a''_1, \dots, a''_j, m).$$

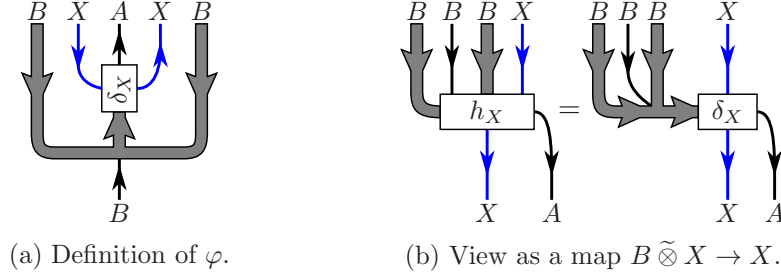
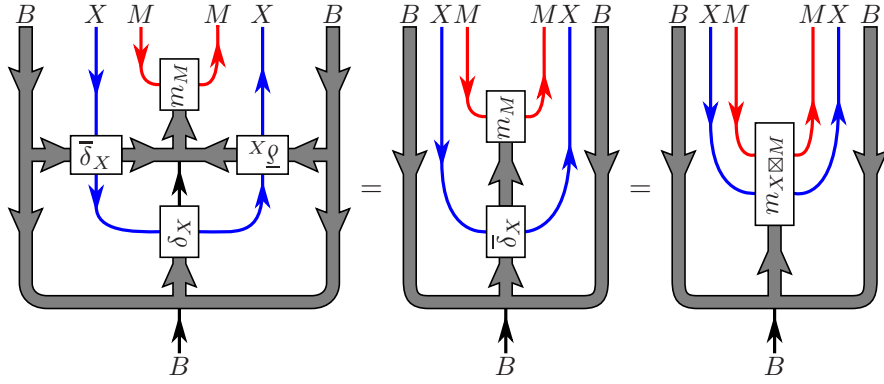
Thus we can build the null-homotopy $\psi \circ h_\varphi + H \circ h_M$ (see Figure 18c). Indeed,

$$\begin{aligned} \partial(\psi \circ h_\varphi) &= \psi \circ \varphi \circ h_M + \psi \circ h_N \circ (\text{id}_A \tilde{\otimes} \varphi), \\ \partial(H \circ h_M) &= \text{id}_M \circ h_M + \psi \circ \varphi \circ h_M. \end{aligned}$$

Alternatively, we can express the null-homotopy of the expression (3) directly as in Figure 18d. \square

Proof of Proposition 4.4. Recall the statement of Proposition 4.4. We are given two differential graded algebras A and B , and three modules ${}_B X^A$, ${}_A Y^B$, and ${}_A M$. We assume that there are homotopy equivalences $X \boxtimes Y \simeq {}_B \mathbb{I}^B$ and $Y \boxtimes X \simeq {}_A \mathbb{I}^A$, and that $X \boxtimes A^\vee \boxtimes X^\vee$ and B^\vee have homologies of the same rank.

We have to construct a homotopy equivalence $\varphi_X: X \boxtimes A^\vee \boxtimes X^\vee \rightarrow B^\vee$, and a homotopy $\nabla_{X \boxtimes M} \simeq \varphi_X \circ (\text{id}_X \boxtimes \nabla_M \boxtimes \text{id}_{X^\vee})$.

FIGURE 19. Two views of the homotopy equivalence φ from Eq. (5).FIGURE 20. Equality of the direct and induced ∇ maps for $X \boxtimes M$.

We start by constructing the morphism φ . We can define it by the following equation:

$$(5) \quad \langle (\varphi_X)_{i|1|j}(b_1, \dots, b_i, (x, a^\vee, x^\vee), b'_1, \dots, b'_j), b'' \rangle \\ = \langle \delta_{i+j+1|1|1}(b'_1, \dots, b'_j, b'', b_1, \dots, b_i, x), (x', a)^\vee \rangle.$$

Again, it is useful to “turn it sideways”. We can reinterpret φ_X as a morphism of type-AD modules $B \tilde{\otimes} X \rightarrow X$. In fact, it is precisely the canonical homotopy equivalence h_X between the two. Diagrams for φ_X and h_X are shown in Figure 19. Since the h_X is a homomorphism, it follows that φ_X is one as well.

Next we show that $\nabla_{X \boxtimes M}$ is homotopic to $\varphi_X \circ (\text{id}_X \boxtimes \nabla_M \boxtimes \text{id}_{X^\vee})$. They are in fact equal. This is best seen in Figure 20. We use the fact that $\bar{\delta}_X$ and δ_X commute with merges and splits.

Finally, we need to show that φ_X is a homotopy equivalence. We will do that by constructing a right homotopy inverse for it. Combined with the fact that the homologies of the two sides have equal rank, this is enough to ascertain that it is indeed a homotopy equivalence.

Recall that $X \boxtimes Y \simeq \mathbb{I}$. Thus there exist morphisms of type-AD B, B -bimodules $f: \mathbb{I} \rightarrow X \boxtimes Y$, and $g: X \boxtimes Y \rightarrow \mathbb{I}$, and a null-homotopy $H: \mathbb{I} \rightarrow \mathbb{I}$

of $\text{id}_{\mathbb{I}} - g \circ f$. Note that $g^\vee: \mathbb{I}^\vee \rightarrow Y^\vee \boxtimes X^\vee$ is a map of type- DA -modules, and $({}_B\mathbb{I}^B)^\vee = {}^B\mathbb{I}_B$.

Let $\varphi_Y: Y \boxtimes B^\vee \boxtimes Y^\vee \rightarrow A$ be defined analogous to φ_X . Construct the homomorphism

$$\begin{aligned} \psi &= (\text{id}_X \boxtimes \varphi_Y \boxtimes \text{id}_{X^\vee}) \circ (f \boxtimes \text{id}_{B^\vee} \boxtimes \text{id}_{Y^\vee} \boxtimes \text{id}_{X^\vee}) \circ (\text{id}_{\mathbb{I}} \boxtimes \text{id}_B \boxtimes g^\vee): \\ &\quad \mathbb{I} \boxtimes B^\vee \boxtimes \mathbb{I} \rightarrow X \boxtimes A^\vee \boxtimes X^\vee. \end{aligned}$$

We need to show that $\varphi_X \circ \psi$ is homotopic to id_{B^\vee} , or more precisely to the canonical isomorphism $\iota: \mathbb{I} \boxtimes B^\vee \boxtimes \mathbb{I} \rightarrow B^\vee$. A graphical representation of $\varphi_X \circ \psi$ is shown in Figure 21a. It simplifies significantly, due to the fact that B is a DG-algebra, and μ_B only has two nonzero terms. The simplified version of $\varphi_X \circ \psi$ is shown in Figure 21b. As usual, it helps to turn the diagram sideways. We can view it as a homomorphism $B \widetilde{\otimes} \mathbb{I} \rightarrow \mathbb{I}$ of type- AD B, B -bimodules. As can be seen from Figure 21c, we get the composition

$$(6) \quad g \circ (h_X \boxtimes \text{id}_Y) \circ (\text{id}_B \widetilde{\otimes} f) = g \circ h_{X \boxtimes Y} \circ (\text{id}_B \widetilde{\otimes} f): B \widetilde{\otimes} \mathbb{I} \rightarrow \mathbb{I}.$$

On the other hand, the homomorphism $\iota: \mathbb{I} \boxtimes B^\vee \boxtimes \mathbb{I} \rightarrow B^\vee$, if written sideways, becomes the homotopy equivalence $h_{\mathbb{I}}: B \widetilde{\otimes} \mathbb{I} \rightarrow \mathbb{I}$. See Figure 22 for the calculation. In the second step we use some new notation. The caps on the thick strands denote a map $\text{Bar } B \rightarrow K$ to the ground ring, which is the identity on $B^{\otimes 0}$, and zero on $B^{\otimes i}$ for any $i > 0$. The dots on the \mathbb{I} strands denote the canonical isomorphism of $\mathbb{I} \boxtimes B^\vee \boxtimes \mathbb{I}$ and B^\vee as modules over the ground ring.

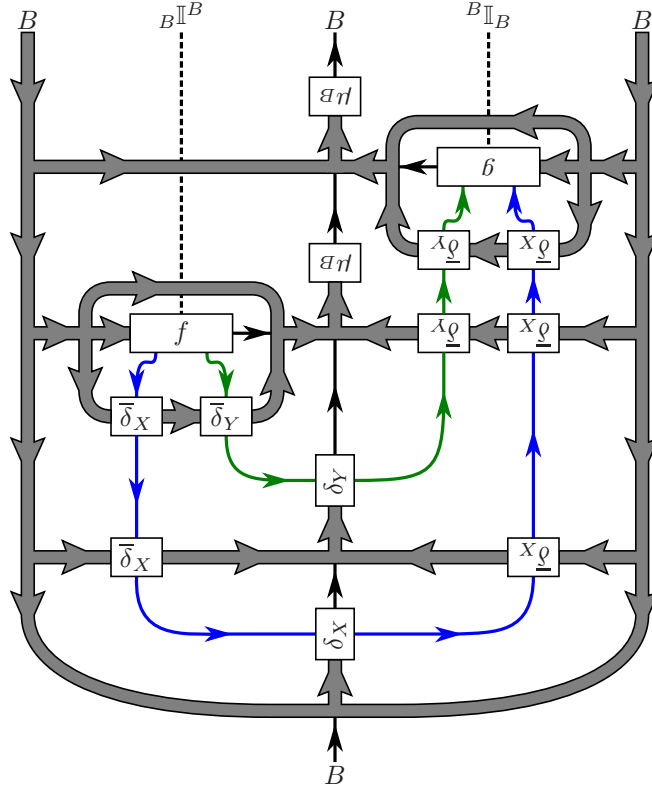
Finding a null-homotopy for $\iota + \varphi_X \circ \psi$ is equivalent to finding a null-homotopy $B \widetilde{\otimes} \mathbb{I} \rightarrow \mathbb{I}$ of $h_{\mathbb{I}} + g \circ h_{X \boxtimes Y} \circ (\text{id}_B \widetilde{\otimes} f)$. There is a null-homotopy $\zeta_f: B \widetilde{\otimes} \mathbb{I} \rightarrow B \widetilde{\otimes} X \boxtimes Y$ of $f \circ h_{\mathbb{I}} + h_{X \boxtimes Y} \circ (\text{id}_B \widetilde{\otimes} f)$. Recall that H was a null-homotopy of $\text{id}_{\mathbb{I}} + g \circ f$. Thus we have

$$\begin{aligned} \partial(H \circ h_{\mathbb{I}} + g \circ \zeta_f) &= (\text{id}_{\mathbb{I}} \circ h_{\mathbb{I}} + g \circ f \circ h_{\mathbb{I}}) \\ &\quad + (g \circ f \circ h_{\mathbb{I}} + g \circ h_{X \boxtimes Y} \circ (\text{id}_B \widetilde{\otimes} F)) \\ &= h_{\mathbb{I}} + g \circ h_{X \boxtimes Y} \circ (\text{id}_B \widetilde{\otimes} F), \end{aligned}$$

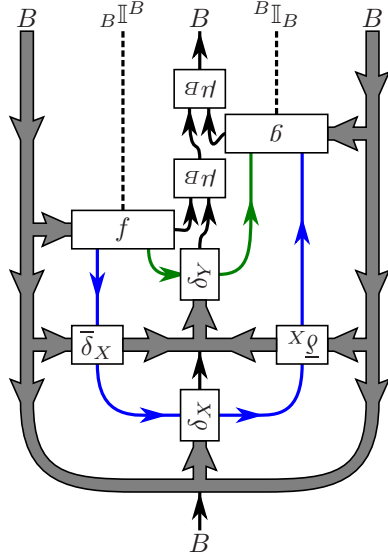
giving us the required null-homotopy.

To finish the proof, notice that if $\varphi_X \circ \psi$ is homotopic to id_B , then it is a quasi-isomorphism, i.e. a homomorphism whose scalar component is a quasi-isomorphism of chain complexes. Moreover, when working with $\mathbb{Z}/2$ -coefficients, as we do, quasi-isomorphisms of \mathcal{A}_∞ -modules and bimodules coincide with homotopy equivalences.

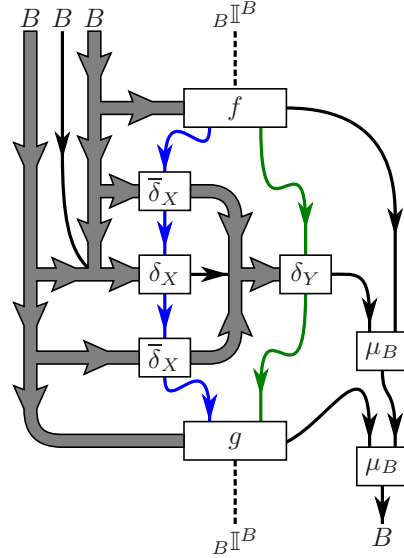
In particular we have that $(\varphi_X \circ \psi)_{0|1|0} = (\varphi_X)_{0|1|0} \circ \psi_{0|1|0}$ induces an isomorphism on homology (in this case the identity map on homology). In particular ψ induces an injection, while φ_X induces a surjection. Combined with the initial assumption that B^\vee and $X \boxtimes A^\vee \boxtimes X^\vee$ have homologies of equal rank, this implies that $(\varphi_X)_{0|1|0}$ and $\psi_{0|1|0}$ induce isomorphisms on homology. That is, φ_X and ψ are quasi-isomorphisms, and so homotopy



(a) Before simplification.

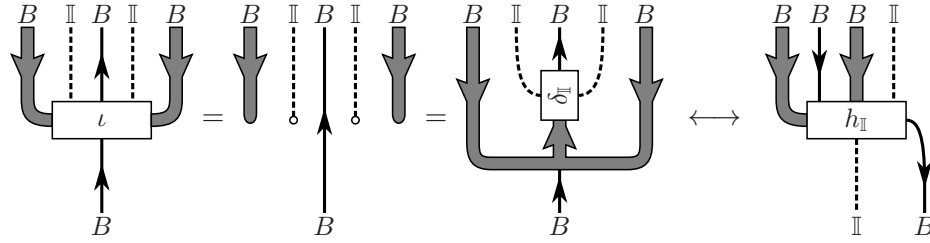


(b) After simplification.



(c) Written sideways.

FIGURE 21. Three views of $\varphi_X \circ \psi: \mathbb{I} \boxtimes B^\vee \boxtimes \mathbb{I} \rightarrow B^\vee$.

FIGURE 22. The equivalence of the morphism ι and $h_{\mathbb{I}}$.

equivalences. This concludes the proof of Proposition 4.4, and with it, of Theorem 4.6. \square

5. PROPERTIES OF THE JOIN MAP

In this section we give some formulas for the join and gluing maps, and prove their formal properties.

5.1. Explicit formulas. We have abstractly defined the join map $\Psi_{\mathcal{W}}$ in terms of $\nabla_{\widehat{BSA}(\mathcal{W})}$ but so far have not given any explicit formula for it. Here we give the general formula, as well as some special cases which are somewhat simpler.

If we want to compute $\Psi_{\mathcal{W}}$ for the join $\mathcal{Y}_1 \uplus_{\mathcal{W}} \mathcal{Y}_2$, we need to pick a parametrization by an arc diagram \mathcal{Z} , with associated algebra A , and representatives U for $\widehat{BSD}(\mathcal{Y}_1)^A$, V for ${}^A\widehat{BSD}(\mathcal{Y}_2)$, and M for ${}^A\widehat{BSA}(\mathcal{W})$. Then we know $SFC(\mathcal{Y}_1) = U \boxtimes M$, $SFC(\mathcal{Y}_2) = M^\vee \boxtimes V$, and $SFC(\mathcal{Y}_1 \uplus_{\mathcal{W}} \mathcal{Y}_2) = U \boxtimes A^\vee \boxtimes V$. As given in Definition 4.5, the join map $\Psi_{\mathcal{W}}$ is

$$\Psi_{\mathcal{W}} = \text{id}_U \boxtimes \nabla_M \boxtimes \text{id}_V : U \boxtimes M \otimes M^\vee \boxtimes V \rightarrow U \boxtimes A^\vee \boxtimes V.$$

In graphic form this can be seen in Figure 23a.

This general form is not good for computations, especially if we try to write it algebraically. However $\Psi_{\mathcal{W}}$ has a much simpler form when M is a *DG-type* module.

Definition 5.1. An \mathcal{A}_∞ -module M_A is of *DG-type* if it is a *DG-module*, i.e., if its structure maps $m_{1|i}$ vanish for $i \geq 2$. A bimodule ${}_A M_B$ is of *DG-type* if $m_{i|1|j}$ vanish, unless (i, j) is one of $(0, 0)$, $(1, 0)$ or $(0, 1)$ (i.e. it is a *DG-module* over $A \otimes B$).

A type-DA bimodule ${}_A M_B$ is of *DG-type* if $\delta_{1|1|j}$ vanish for all $j \geq 2$. A type-DD bimodule ${}_A M^B$ is of *DG-type* if $\delta_{1|1|1}(x)$ is always in $A \otimes X \otimes 1 + 1 \otimes X \otimes B$ (i.e. it is separated). All type *D*-modules M^A are *DG-type*.

The \boxtimes -product of any combination of *DG-type* modules is also *DG-type*. All modules \widehat{BSA} , \widehat{BSD} , \widehat{BSAA} , etc., computed from a nice diagram are of *DG-type*.

Proposition 5.2. *Let the manifolds \mathcal{Y}_1 , \mathcal{Y}_2 , and \mathcal{W} , and the modules U , V , and M be as in the above discussion. If M is DG-type, the formula for the join map $\Psi_{\mathcal{W}}$ simplifies to:*

$$(7) \quad \Psi_{\mathcal{W}}(u \boxtimes m \otimes n^\vee \boxtimes v) = \sum_a \langle m_M(a, m), n^\vee \rangle \cdot u \boxtimes a^\vee \boxtimes v,$$

where the sum is over a $\mathbb{Z}/2$ -basis for A . A graphical representation is given in Figure 23b.

Finally, an even simpler case is that of elementary modules. We will see later that elementary modules play an important role for gluing, and for the relationship between the bordered and sutured theories.

Definition 5.3. *A type-A module ${}_A M$ (or similarly M_A) is called elementary if the following conditions hold:*

- (1) *M is generated by a single element m over $\mathbb{Z}/2$.*
- (2) *All structural operations on M vanish (except for multiplication by an idempotent, which might be identity).*

A type-D module ${}^A M$ (or M^A), is called elementary if the following conditions hold:

- (1) *M is generated by a single element m over $\mathbb{Z}/2$.*
- (2) *$\delta(m) = 0$.*

Notice that for an elementary module $M = \{0, m\}$ we can decompose m as a sum $m = \iota_1 m + \dots + \iota_k m$, where (ι_i) is the canonical basis of the ground ring. Thus we must have $\iota_i m = m$ for some i , and $\iota_j m = 0$ for all $i \neq j$. Therefore, elementary (left) modules over A are in a 1-to-1 correspondence with the canonical basis for its ground ring.

We only use elementary type-A modules in this section but we will need both types later.

Remark. For the algebras we discuss, the elementary type-A modules are precisely the simple modules. The elementary type-D modules are the those ${}^A M$ for which $A \boxtimes M \in {}_A \text{Mod}$ is an elementary projective module.

Proposition 5.4. *If ${}_A M = \{m, 0\}$ is an elementary module corresponding to the basis idempotent ι_M , then the join map $\Psi_{\mathcal{W}}$ reduces to*

$$(8) \quad \Psi_{\mathcal{W}}(u \boxtimes m \otimes m^\vee \boxtimes v) = u \boxtimes \iota_M^\vee \boxtimes v.$$

Graphically, this is given in Figure 23c.

Moreover, in this case, $SFC(\mathcal{Y}_1) = U \boxtimes M \cong U \cdot \iota_M \subset U$ and $SFC(\mathcal{Y}_2) = M \boxtimes V \cong \iota_M \cdot V \subset V$ as chain complexes.

Proposition 5.2 and Proposition 5.4 follow directly from the definitions of DG-type and elementary modules.

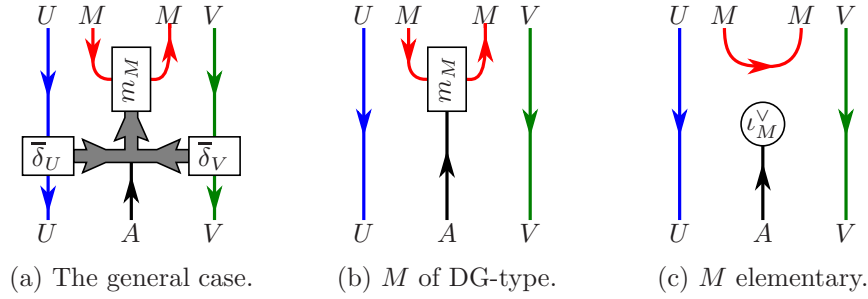


FIGURE 23. Full expression for join map in three cases.

5.2. Formal properties. In this section we will show that the join map has the formal properties stated in Theorem 1. A more precise statement of the properties is given below.

Theorem 5.5. *The following properties hold:*

- (1) *Let \mathcal{Y}_1 and \mathcal{Y}_2 be sutured and \mathcal{W} be partially sutured, with embeddings $\mathcal{W} \hookrightarrow \mathcal{Y}_1$ and $-\mathcal{W} \hookrightarrow \mathcal{Y}_2$. There are natural identifications of the disjoint unions $\mathcal{Y}_1 \sqcup \mathcal{Y}_2$ and $\mathcal{Y}_2 \sqcup \mathcal{Y}_1$, and of the joins $\mathcal{Y}_1 \mathbin{\mathbb{U}}_{\mathcal{W}} \mathcal{Y}_2$ and $\mathcal{Y}_2 \mathbin{\mathbb{U}}_{-\mathcal{W}} \mathcal{Y}_1$. Under this identification, there is a homotopy*

$$\Psi_{\mathcal{W}} \simeq \Psi_{-\mathcal{W}}.$$

- (2) *Let \mathcal{Y}_1 , \mathcal{Y}_2 , and \mathcal{Y}_3 be sutured, and \mathcal{W}_1 and \mathcal{W}_2 be partially sutured, such that there are embeddings $\mathcal{W}_1 \hookrightarrow \mathcal{Y}_1$, $(-\mathcal{W}_1 \sqcup \mathcal{W}_2) \hookrightarrow \mathcal{Y}_2$, and $-\mathcal{W}_2 \hookrightarrow \mathcal{Y}_3$. The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} SFC(\mathcal{Y}_1 \sqcup \mathcal{Y}_2 \sqcup \mathcal{Y}_3) & \xrightarrow{\Psi_{\mathcal{W}_1}} & SFC(\mathcal{Y}_1 \mathbin{\mathbb{U}} \mathcal{Y}_2 \sqcup \mathcal{Y}_3) \\ \Psi_{\mathcal{W}_2} \downarrow & \searrow \Psi_{\mathcal{W}_1 \sqcup -\mathcal{W}_2} & \downarrow \Psi_{\mathcal{W}_2} \\ SFC(\mathcal{Y}_1 \sqcup \mathcal{Y}_2 \mathbin{\mathbb{U}} \mathcal{Y}_3) & \xrightarrow{\Psi_{\mathcal{W}_1}} & SFC(\mathcal{Y}_1 \mathbin{\mathbb{U}} \mathcal{Y}_2 \mathbin{\mathbb{U}} \mathcal{Y}_3) \end{array}$$

- (3) *Let \mathcal{W} be partially sutured. There is a canonical element $[\Delta_{\mathcal{W}}]$ in the sutured Floer homology $SFH(\mathcal{D}(\mathcal{W}))$ of the double of \mathcal{W} . If Δ is any representative for $[\Delta_{\mathcal{W}}]$, and there is an embedding $\mathcal{W} \hookrightarrow \mathcal{Y}$, then*

$$(9) \quad \Psi_{\mathcal{W}}(\cdot, \Delta) \simeq \text{id}_{SFC(\mathcal{Y})}: SFC(\mathcal{Y}) \rightarrow SFC(\mathcal{Y}).$$

Proof. We will prove the three parts in order.

For part (1), take representatives U^A for $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W})$, ${}^A V$ for $\widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W})$, and ${}_A M$ for $\widehat{BSA}(\mathcal{W})$. The main observation here is that we can turn left modules into right modules and vice versa, by reflecting all diagrams along the vertical axis (see Appendix B.6). If we reflect the entire diagram for $\Psi_{\mathcal{M}}$, domain and target chain complexes are turned into isomorphic ones and we get a new map that is equivalent.

The domain $U^A \boxtimes {}_A M \otimes M^\vee {}_A \boxtimes {}^A V$ becomes $V^{A^{\text{op}}} \boxtimes {}_{A^{\text{op}}} M^\vee \otimes M_{A^{\text{op}}} \boxtimes {}^{A^{\text{op}}} U$, and the target $U^A \boxtimes {}_A A^\vee {}_A \boxtimes {}^A V$ becomes $V^{A^{\text{op}}} \boxtimes {}_{A^{\text{op}}} (A^\vee)^{\text{op}} {}_{A^{\text{op}}} \boxtimes {}^{A^{\text{op}}} U$.

Notice that $V^{A^{\text{op}}}$ is $\widehat{BSD}(\mathcal{Y}_2 \setminus -\mathcal{W})$, ${}_{A^{\text{op}}}U$ is $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W})$, and ${}_{A^{\text{op}}}M^\vee$ is $\widehat{BSA}(-\mathcal{W})$. In addition $(A^\vee)^{\text{op}} = (A^{\text{op}})^\vee$. Since the map ∇_M is completely symmetric, when we reflect it, we get ∇_{M^\vee} . Everything else is preserved, so reflecting $\Psi_{\mathcal{W}}$ gives precisely $\Psi_{-\mathcal{W}}$. This finishes part (1).

For part (2), the equivalence is best seen by working with convenient representatives. Pick the following modules as representatives: U^A for $\widehat{BSD}(\mathcal{Y}_1 \setminus \mathcal{W}_1)$, ${}^AX^B$ for $\widehat{BSDD}(\mathcal{Y}_2 \setminus (-\mathcal{W}_1 \cup \mathcal{W}_2))$, BV for $\widehat{BSD}(\mathcal{Y}_1)$, ${}_AM$ for $\widehat{BSA}(\mathcal{W}_1)$ and ${}_BN$ for $\widehat{BSD}(\mathcal{W}_2)$. We can always choose M , N , and X to be of DG-type in the sense of Definition 5.1. Since X is of DG-type, taking the \boxtimes -product with it is associative. (This is only true up to homotopy in general). Since M and N are DG-type, we can apply Proposition 5.2 to get formulas for $\Psi_{\mathcal{W}_1}$ and $\Psi_{\mathcal{W}_2}$. The two possible compositions are shown in Figures 24a and 24b.

To compute $\Psi_{\mathcal{W}_1 \cup -\mathcal{W}_2}$, notice that $(U \otimes V)^{A, B^{\text{op}}}$ represents $\widehat{BSDD}((\mathcal{Y}_1 \cup \mathcal{Y}_3) \setminus (\mathcal{W}_1 \cup -\mathcal{W}_3))$, ${}^{A, B^{\text{op}}}X$ represents $\widehat{BSDD}(\mathcal{Y}_2 \setminus (-\mathcal{W}_1 \cup \mathcal{W}_2))$, and ${}_{A, B^{\text{op}}}(M \otimes N^\vee)$ is a DG-type module representing $\widehat{BSAA}(\mathcal{W}_1 \cup -\mathcal{W}_2)$. To compute the join map, we need to convert them to single modules. For type-DD modules, this is trivial (any A, B -bimodule is automatically an $A \otimes B$ -module and vice versa). For type-AA modules, this could be complicated in general. Luckily, it is easy for DG-type modules. Indeed, if $P_{A, B}$ is DG-type, the corresponding $A \otimes B$ -module $P_{A \otimes B}$ is also DG-type, with algebra action

$$m_{1|1}(\cdot, a \otimes b) = m_{1|1|0}(\cdot, a) \circ m_{1|0|1}(\cdot, b) = m_{1|0|1}(\cdot, b) \circ m_{1|1|0}(\cdot, a).$$

In the definition of bimodule invariants in [Zar09], the procedure used to get \widehat{BSAA} from \widehat{BSA} , and \widehat{BSDD} from \widehat{BSD} is exactly the reverse of this construction.

Thus, we can see that $(U \otimes V)^{A \otimes B^{\text{op}}}$ represents $\widehat{BSD}((\mathcal{Y}_1 \cup \mathcal{Y}_3) \setminus (\mathcal{W}_1 \cup -\mathcal{W}_3))$, ${}^{A \otimes B^{\text{op}}}X$ represents $\widehat{BSD}(\mathcal{Y}_2 \setminus (-\mathcal{W}_1 \cup \mathcal{W}_2))$, and ${}_{A \otimes B^{\text{op}}}(M \otimes N^\vee)$ represents $\widehat{BSA}(\mathcal{W}_1 \cup -\mathcal{W}_2)$. It is also easy to check that

$${}_AA^\vee \otimes {}_{B^{\text{op}}}(B^{\text{op}})^\vee \cong {}_{A \otimes B^{\text{op}}}(A \otimes B^{\text{op}})^\vee {}_{A \otimes B^{\text{op}}}.$$

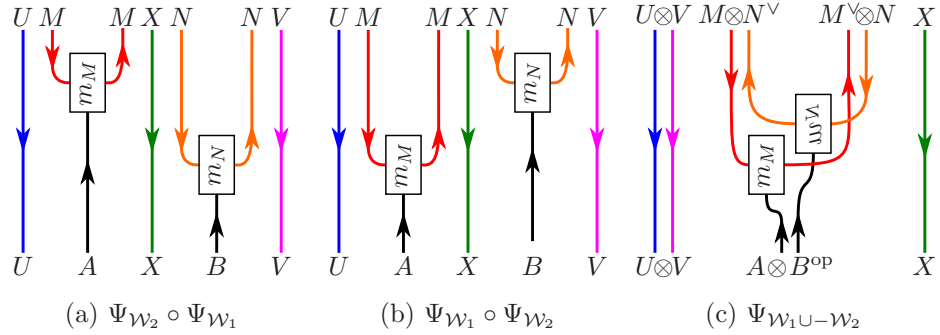
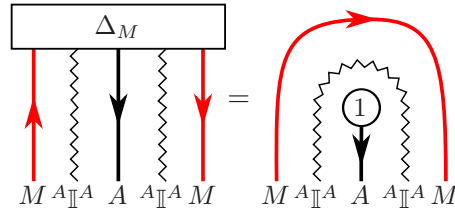
We can see a diagram for $\Psi_{\mathcal{W}_1 \cup -\mathcal{W}_2}$ in Figure 24c. By examining the diagrams, we see that the three maps are the same, which finishes part (2).

Part (3) requires some more work, so we will split it in several steps. We will define Δ_M for a fixed representative M of $\widehat{BSD}(\mathcal{W})$. We will prove that $[\Delta_M]$ does not depend on the choice of M . Finally, we will use a computational lemma to show that Eq. (9) holds for Δ_M .

First we will introduce some notation. Given an \mathcal{A}_∞ -module ${}_AM$ over $A = \mathcal{A}(\mathcal{Z})$, define the *double* of M to be

$$(10) \quad \mathcal{D}(M) = M^\vee \boxtimes ({}^A\mathbb{I}^A \boxtimes A \boxtimes {}^A\mathbb{I}^A) \boxtimes M.$$

Note that if $M = \widehat{BSA}(\mathcal{W})$, then $\mathcal{D}(M) = \widehat{BSA}(-\mathcal{W}) \boxtimes \widehat{BSDD}(\mathcal{T}\mathcal{W}_{\mathcal{F}}, -) \boxtimes \widehat{BSA}(\mathcal{W}) \simeq \text{SFC}(\mathcal{D}(\mathcal{W}))$. Next we define the *diagonal element* $\Delta_M \in \mathcal{D}(M)$

FIGURE 24. Three ways to join \mathcal{Y}_1 , \mathcal{Y}_2 , and \mathcal{Y}_3 .FIGURE 25. The diagonal element Δ_M .

as follows. Pick a basis (m_1, \dots, m_k) of M over $\mathbb{Z}/2$. Define

$$(11) \quad \Delta_M = \sum_{i=1}^k m_i \boxtimes (* \boxtimes 1 \boxtimes *) \boxtimes m_i^\vee.$$

It is easy to check that this definition does not depend on the choice of basis. Indeed there is a really simple diagrammatic representation of Δ_M , given in Figure 25. We think of it as a linear map from $\mathbb{Z}/2$ to $\mathcal{D}(M)$. It is also easy to check that $\partial \Delta_M = 0$. Indeed, writing out the definition of $\partial \Delta_M$, there are only two nonzero terms which cancel.

The proof that $[\Delta_M]$ does not depend on the choices of A and M is very similar to the proof of Theorem 4.6, so will omit it. (It involves showing independence from M , as well as from A via a quasi-invertible bimodule ${}^A X_B$.)

Lemma 5.6. *Let A be a differential graded algebra, coming from an arc diagram \mathcal{Z} . There is a homotopy equivalence*

$$c_A: {}^A \mathbb{I}^A \boxtimes A^\vee \boxtimes {}^A \mathbb{I}^A \boxtimes {}_A A_A \rightarrow {}^A \mathbb{I}_A,$$

given by

$$(c_A)_{1|1|0} (* \boxtimes a^\vee \boxtimes * \boxtimes b) = \begin{cases} b \otimes * & \text{if } a \text{ is an idempotent,} \\ 0 & \text{otherwise.} \end{cases}$$

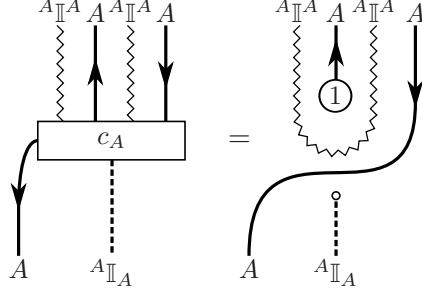


FIGURE 26. The cancellation homotopy equivalence $c_A : \mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I} \boxtimes A \rightarrow \mathbb{I}$.

Here we use $*$ to denote the unique element with compatible idempotents in the two versions of \mathbb{I} . (Both versions have generators in 1-to-1 correspondence with the basis idempotents.)

Remark. As we mentioned earlier, one has to be careful when working with type- DD modules. While \boxtimes and $\tilde{\boxtimes}$ are usually associative by themselves, and with each other, this might fail when a DD -module is involved, in which case we only have associativity up to homotopy equivalence. However, this could be mitigated in two situations. If the DD -module is DG-type (which fails for $A^I A$), or if the type- A modules on both sides are DG-type, then true associativity still holds. This is true for A and A^\vee , so the statement of the lemma makes sense.

Proof of Lemma 5.6. Note that we can easily see that there is *some* homotopy equivalence $(\mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I}) \boxtimes A \simeq \mathbb{I}$, since the left-hand side is

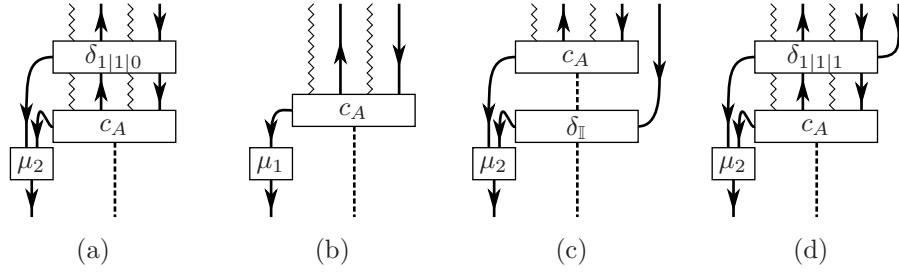
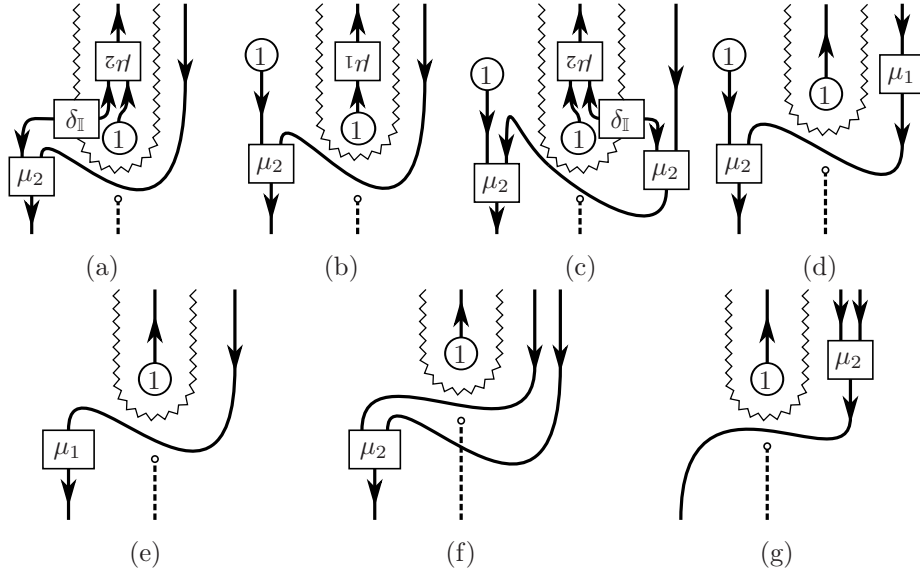
$$\widehat{BSDD}(\mathcal{TW}_{\mathcal{F},+}) \boxtimes \widehat{BSAA}(\mathcal{TW}_{-\mathcal{F},-}) \simeq \widehat{BSDA}(\mathcal{TW}_{\mathcal{F},+} \cup \mathcal{TW}_{-\mathcal{F},-}),$$

while the right side is $\widehat{BSDA}(\mathcal{F} \times [0, 1])$, and those bordered sutured manifolds are the same. The difficulty is in finding the precise homotopy equivalence, which we need for computations, in order to “cancel” A^\vee and A .

First, we need to show that c_A is a homomorphism. This is best done graphically. The definition of c_A is represented in Figure 26. The notation we use there is that $A^I A$ is a jagged line, without a direction, since \mathbb{I} is its own dual. A^I_A is represented by a dashed line. As before the line can start or end with a dot, signifying the canonical isomorphism given by $\cdot \boxtimes *$.

We need to show that $\partial c_A = 0$. Note that by definition c_A only has a $1|1|0$ -term. On the other hand δ on $\mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I} \boxtimes A$ has only $1|1|0$ - and $1|1|1$ -terms, while δ on \mathbb{I} has only a $1|1|1$ -term.

Thus only four terms from the definition of ∂c_A survive. These are shown in Figure 27. Expanding the definition of δ on $\mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I} \boxtimes A$ in terms of the operations of \mathbb{I} , A , and A^\vee , we get seven terms. We can see them in Figure 28. The terms in Figures 28a—28d correspond to Figure 27a, while those in Figures 28e—28g correspond to Figures 27b—27d, respectively. Six of the terms cancel in pairs, while the one in 28b equals 0.

FIGURE 27. Nontrivial terms of ∂c_A .FIGURE 28. Elementary terms of ∂h .

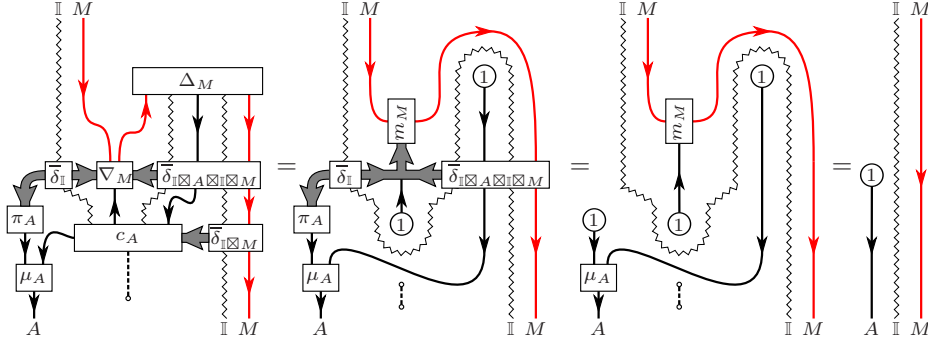
Showing that c_A is a homotopy equivalence is somewhat roundabout. First we will show that the induced map

$$\mathrm{id}_A \boxtimes c_A: A \boxtimes ({}^A\mathbb{I}^A \boxtimes A^\vee \boxtimes {}^A\mathbb{I}^A \boxtimes A) \rightarrow A \boxtimes {}^A\mathbb{I}_A \cong A$$

is a homotopy equivalence. It is easy to see that the map is

$$(\mathrm{id}_A \boxtimes c_A)_{0|1|0}(a \boxtimes * \boxtimes b^\vee \boxtimes * \boxtimes c) = \begin{cases} a \cdot c & \text{if } b \text{ is an idempotent,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it is surjective. Indeed, $\mathrm{id}_A \boxtimes c_A(a \boxtimes * \boxtimes 1^\vee \boxtimes * \boxtimes 1) = a$ for all $a \in A$. Thus the induced map on homology is surjective. But the source and domain are homotopy equivalent for topological reasons (both represent $\widehat{BSAA}(\mathcal{TW}_{\overline{\mathcal{F}}, -})$). This implies that $\mathrm{id}_A \boxtimes c_A$ is a quasi-isomorphism, and a homotopy equivalence. But $(\mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I}) \boxtimes A \simeq \mathbb{I}$ and $A \boxtimes (\mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I}) \simeq \mathbb{I}$ for topological reasons, so $A \boxtimes \cdot$ is an equivalence of derived categories. Thus,

FIGURE 29. Proof that $\Psi_M(\cdot, \Delta_M) \simeq \text{id}$.

c_A itself must have been a homotopy equivalence, which finishes the proof of the lemma. \square

We will now use Lemma 5.6, to show that for any \mathcal{Y} there is a homotopy $\Psi_{\mathcal{W}}(\cdot, \Delta_M) \simeq \text{id}_{SFC(\mathcal{Y})}$. Let c_A be the homotopy equivalence from the lemma. There is a sequence of homomorphisms as follows.

$$\begin{aligned}
 & \mathbb{I} \boxtimes M \\
 & \downarrow \text{id}_{\mathbb{I} \boxtimes M} \otimes \Delta_M \\
 & \mathbb{I} \boxtimes M \otimes \mathcal{D}(M) \cong \mathbb{I} \boxtimes M \otimes M^\vee \boxtimes \mathbb{I} \boxtimes A \boxtimes \mathbb{I} \boxtimes M \\
 & \downarrow \text{id}_{\mathbb{I}} \boxtimes \nabla_M \boxtimes \text{id}_{\mathbb{I} \boxtimes A \boxtimes \mathbb{I} \boxtimes M} \\
 & \mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I} \boxtimes A \boxtimes \mathbb{I} \boxtimes M \\
 & \downarrow c_A \boxtimes \text{id}_{\mathbb{I} \boxtimes M} \\
 & \mathbb{I} \boxtimes M
 \end{aligned}$$

The composition of these maps is shown in Figure 29. As we can see from the diagram, it is equal to $\text{id}_{\mathbb{I}} \boxtimes \text{id}_M$. If $U = \widehat{BSD}(\mathcal{Y} \setminus \mathcal{W})$, then by applying the functor $\text{id}_U \boxtimes \cdot$ to both homomorphisms, we see that

$$(\text{id}_U \boxtimes c_A \boxtimes \text{id}_{\mathbb{I} \boxtimes M}) \circ \Psi_M \circ (\text{id}_{SFC(\mathcal{Y})} \boxtimes \Delta_M) = \text{id}_{SFC(\mathcal{Y})},$$

which is equivalent to Eq. (9). \square

5.3. Self-join and self-gluing. So far we have been talking about the join or gluing of two disjoint sutured manifolds. However, we can extend these notions to a self-join or self-gluing of a single manifold. For example if there is an embedding $(\mathcal{W} \sqcup -\mathcal{W}) \hookrightarrow \mathcal{Y}$, then we can define the *self-join* of \mathcal{Y} along \mathcal{W} to be the concatenation

$$\mathcal{Y} \mathfrak{U}_{\mathcal{W}, \circ} = (Y \setminus (\mathcal{W} \sqcup -\mathcal{W})) \cup_{\mathcal{F} \cup \overline{\mathcal{F}}} \mathcal{TW}_{\mathcal{F}, +} \cong \mathcal{Y} \mathfrak{U}_{\mathcal{W} \sqcup -\mathcal{W}} \mathcal{D}(\mathcal{W}).$$

It is easy to see that if \mathcal{W} and $-\mathcal{W}$ embed into different components of \mathcal{Y} , this is the same as the regular join.

Similarly, we can extend the join map to a *self-join map*

$$\Psi_{\mathcal{W}, \circ}: SFC(\mathcal{Y}) \rightarrow SFC(\mathcal{Y} \uplus_{\mathcal{W} \sqcup -\mathcal{W}} \mathcal{D}(\mathcal{W})) \simeq SFC(\mathcal{Y} \uplus_{\mathcal{W}, \circ}),$$

by setting

$$\Psi_{\mathcal{W}, \circ} = \Psi_{\mathcal{W} \sqcup -\mathcal{W}}(\cdot, \Delta_{\mathcal{W}}).$$

Again, if \mathcal{W} and $-\mathcal{W}$ embed into disjoint components of \mathcal{Y} , $\Psi_{\mathcal{W}, \circ}$ is, up to homotopy, the same as the regular join map $\Psi_{\mathcal{W}}$. This follows quickly from properties (2) and (3) in Theorem 5.5.

6. THE BORDERED INVARIANTS IN TERMS OF SFH

In this section we give a (partial) reinterpretation of bordered and bordered sutured invariants in terms of SFH and the gluing map Ψ . This is a more detailed version of Theorem 2.

6.1. Elementary dividing sets. Recall Definition 2.3 of a dividing set. Suppose we have a sutured surface $\mathcal{F} = (F, \Lambda)$ parametrized by an arc diagram $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ of rank k . We will define a set of 2^k distinguished dividing sets.

Before we do that, recall the way an arc diagram parametrizes a sutured surface, from Section 3.1. There is an embedding of the graph $G(\mathcal{Z})$ into F , such that $\partial \mathbf{Z} = \Lambda$ (Recall Figure 8). We will first define the elementary dividing sets in the cases that \mathcal{Z} is of α -type. In that case the image of \mathbf{Z} is a push-off of S_+ into the interior of F . Denote the regions between S_+ and \mathbf{Z} by R_0 . It is a collection of discs, one for each component of S_+ . The images of the arcs $e_i \subset G(\mathcal{Z})$ are in the complement $F \setminus R_0$.

Definition 6.1. *Let $I \subset \{1, \dots, k\}$. The elementary dividing set for \mathcal{F} associated to I is the dividing set Γ_I constructed as follows. Let R_0 be the region defined above. Set*

$$R_+ = R_0 \cup \bigcup_{i \in I} \nu(e_i) \subset F.$$

Then $\Gamma_I = (\partial R_+) \setminus S_+$.

If \mathcal{Z} is of β -type, repeat the same procedure, substituting R_- for R_+ and S_- for S_+ . For example the region R_0 consists of discs bounded by $S_- \cup \mathbf{Z}$. Examples of both cases are given in Figure 30.

We refer to the collection of Γ_I for all 2^k -many subsets of $\{1, \dots, k\}$ as *elementary dividing sets for \mathcal{Z}* . The reason they are important is the following proposition.

Proposition 6.2. *Let \mathcal{Z} be an arc diagram of rank k , and let $I \subset \{1, \dots, k\}$ be any subset. Let ι_I be the idempotent for $A = \mathcal{A}(\mathcal{Z})$ corresponding to horizontal arcs at all $i \in I$, and let ι_{I^c} be the idempotent corresponding to the complement of I . Let \mathcal{W}_I be the cap associated to the elementary dividing set Γ_I .*

Then the following hold:

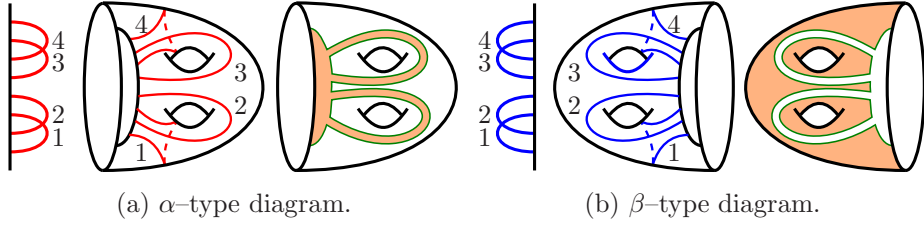


FIGURE 30. Elementary dividing sets for an arc diagram. In each case we show the arc diagram, its embedding into the surface, and the dividing set $\Gamma_{\{2,3\}}$. The shaded regions are R_+ .

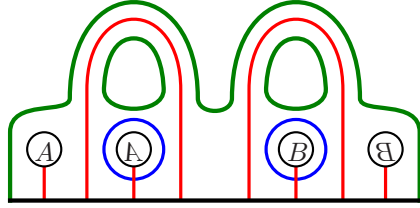


FIGURE 31. Heegaard diagram \mathcal{H} for the cap $\mathcal{W}_{2,3}$ corresponding to the dividing set from Figure 30a.

- ${}^A\widehat{BSD}(\mathcal{W})$ is (represented by) the elementary type-D module for ι_I .
- ${}_A\widehat{BSA}(\mathcal{W})$ is (represented by) the elementary type-A module for ι_{I^c} .

Proof. The key fact is that there is a particularly simple Heegaard diagram \mathcal{H} for \mathcal{W}_I . For concreteness we will assume \mathcal{Z} is a type- α diagram, though the case of a type- β diagram is completely analogous.

The diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \mathcal{Z})$ contains no α -circles, exactly one α -arc α_i^a for each matched pair $M^{-1}(i)$, and $k - \#I$ many β -circles. Each β -circle has exactly one intersection point on it, with one of α_i^a , for $i \notin I$. This implies that there is exactly one generator $\mathbf{x} \in \mathcal{G}(\mathcal{H})$, that occupies the arcs for I^c . This implies that $\widehat{BSD}(\mathcal{W}_I)$ and $\widehat{BSA}(\mathcal{W}_I)$ are both $\{\mathbf{x}, 0\}$ as $\mathbb{Z}/2$ -modules. The actions of the ground ring are $\iota_I \cdot \mathbf{x} = \mathbf{x}$ for $\widehat{BSD}(\mathcal{W}_I)$ and $\iota_{I^c} \cdot \mathbf{x} = \mathbf{x}$ for $\widehat{BSA}(\mathcal{W}_I)$. This was one of the two requirements for an elementary module.

The connected components of $\Sigma \setminus (\alpha \cup \beta)$ are in 1-to-1 correspondence with components of ∂R_+ . In fact each such region is adjacent to exactly one component of $\partial \Sigma \setminus \mathbf{Z}$. Therefore, there are only boundary regions and no holomorphic curves are counted for the definitions of $\widehat{BSD}(\mathcal{W}_I)$ and $\widehat{BSA}(\mathcal{W}_I)$. This was the other requirement for an elementary module, so the proof is complete. The diagram \mathcal{H} can be seen in Figure 31. \square

We will define one more type of object. Let \mathcal{F} be a sutured surface parametrized by some arc diagram \mathcal{Z} . Let I and J be two subsets of $\{1, \dots, k\}$. Consider the sutured manifold $-\mathcal{W}_I \cup \mathcal{TW}_{-\mathcal{F}, -} \cup \mathcal{W}_J$. Since $-\mathcal{W}_I$ and \mathcal{W}_J are caps, topologically this is $F \times [0, 1]$. The dividing set can

be described as follows. Along $F \times \{0\}$ it is $\Gamma_I \times \{0\}$, along $F \times \{1\}$ it is $\Gamma_J \times \{1\}$, and along $\partial F \times [0, 1]$ it consists of arcs in the $[0, 1]$ direction with a partial negative twist.

Definition 6.3. Let $\Gamma_{I \rightarrow J}$ denote the dividing set on $\partial(F \times [0, 1])$, such that

$$(F \times [0, 1], \Gamma_{I \rightarrow J}) = -\mathcal{W}_I \cup \mathcal{TW}_{\overline{\mathcal{F}}, -} \cup \mathcal{W}_J.$$

6.2. Main results. The main results of this section are the following two theorems. We will give the proofs in the next subsection.

Theorem 6.4. Let \mathcal{F} be a sutured surface parametrized by an arc diagram \mathcal{Z} . The homology of $A = \mathcal{A}(\mathcal{Z})$ decomposes as the sum

$$(12) \quad H_*(A) = \bigoplus_{I, J \subset \{1, \dots, k\}} \iota_I \cdot H_*(A) \cdot \iota_J = \bigoplus_{I, J \subset \{1, \dots, k\}} H_*(\iota_I \cdot A \cdot \iota_J),$$

where

$$(13) \quad \iota_I \cdot H_*(A) \cdot \iota_J \cong SFH(F \times [0, 1], \Gamma_{I \rightarrow J}).$$

Multiplication μ_2 descends to homology as

$$(14) \quad \begin{aligned} \mu_H = \Psi_{(F, \Gamma_J)}: SFH(F \times [0, 1], \Gamma_{I \rightarrow J}) \otimes SFH(F \times [0, 1], \Gamma_{J \rightarrow K}) \\ \rightarrow SFH(F \times [0, 1], \Gamma_{I \rightarrow K}), \end{aligned}$$

and is 0 on all other summands.

Theorem 6.5. Let $\mathcal{Y} = (Y, \Gamma, \mathcal{F})$ be a bordered sutured manifold where \mathcal{F} parametrized by \mathcal{Z} . Then there is a decomposition

$$(15) \quad \begin{aligned} H_*\left(\widehat{BSA}(Y)_A\right) &= \bigoplus_{I \subset \{1, \dots, k\}} H_*\left(\widehat{BSA}(\mathcal{Y})\right) \cdot \iota_I \\ &= \bigoplus_{I \subset \{1, \dots, k\}} H_*\left(\widehat{BSA}(Y) \cdot \iota_I\right), \end{aligned}$$

where

$$(16) \quad H_*\left(\widehat{BSA}(Y)\right) \cdot \iota_I \cong SFH(Y, \Gamma \cup \Gamma_I).$$

Moreover, the $m_{1|1}$ action of A on \widehat{BSA} descends to the following action on homology:

$$(17) \quad m_H = \Psi_{(F, \Gamma_I)}: SFH(Y, \Gamma \cup \Gamma_I) \otimes SFH(F \times I, \Gamma_{I \rightarrow J}) \rightarrow SFH(Y, \Gamma \cup \Gamma_J),$$

and $m_H = 0$ on all other summands.

Similar statements hold for left A -modules ${}_A\widehat{BSA}(\mathcal{Y})$, and for bimodules ${}_A\widehat{BSAA}(\mathcal{Y})_B$.

Theorem 6.4 and 6.5, give us an alternative way to think about bordered sutured Floer homology, or pure bordered Floer homology. (Recall that as shown in [Zar09], the bordered invariants \widehat{CFD} and \widehat{CFA} are special cases of \widehat{BSD} and \widehat{BSA} .) More remarkably, as we show in [Zar], $H_*(A)$, μ_H , and m_H can be expressed in purely contact-geometric terms.

For practical purposes, A and \widehat{BSA} can be replaced by the \mathcal{A}_∞ -algebra $H_*(A)$ and the \mathcal{A}_∞ -module $H_*(\widehat{BSA})$ over it. For example, the pairing theorem will still hold. This is due to the fact that (using $\mathbb{Z}/2$ -coefficients), an \mathcal{A}_∞ -algebra or module is always homotopy equivalent to its homology.

We would need, however, the higher multiplication maps of $H_*(A)$, and the higher actions of $H_*(A)$ on $H_*(\widehat{BSA})$. The maps μ_H and m_H that we just computed are only single terms of those higher operations. (Even though A is a DG-algebra, $H_*(A)$ usually has nontrivial higher multiplication.)

6.3. Proofs. In this section we prove Theorems 6.4 and 6.5. Since there is a lot of overlap of the two results and the arguments, we will actually give a combined proof of a mix of statements from both theorems. The rest follow as corollary.

Combined proof of Theorem 6.4 and Theorem 6.5. First, note that Eq. (12) and Eq. (15) follow directly from the fact that the idempotents generate the ground ring over $\mathbb{Z}/2$.

We will start by proving a generalization of Eq. (13) and Eq. (16). The statement is as follows. Let \mathcal{F} and \mathcal{F}' be two sutured surfaces parametrized by the arc diagrams \mathcal{Z} and \mathcal{Z}' of rank k and k' , respectively. Let $A = \mathcal{A}(\mathcal{Z})$ and $B = \mathcal{A}(\mathcal{Z}')$. Let $\mathcal{Y} = (Y, \Gamma, \overline{\mathcal{F}} \sqcup \mathcal{F}')$ be a bordered sutured manifold, and let $M = {}_A\widehat{BSAA}(\mathcal{Y})_B$.

Fix $I \subset \{1, \dots, k\}$ and $J \subset \{1, \dots, k'\}$. Let \mathcal{W}_I and \mathcal{W}'_J be the respective caps associated to the dividing sets Γ_I on \mathcal{F} and Γ'_J on \mathcal{F}' . Then the following homotopy equivalence holds.

$$(18) \quad \iota_I \cdot \widehat{BSAA}(\mathcal{Y}) \cdot \iota_J \simeq SFC(Y, \Gamma_I \cup \Gamma \cup \Gamma'_J).$$

The proof is easy. Notice that the sutured manifold $(Y, \Gamma_I \cup \Gamma \cup \Gamma'_J)$ is just $-\mathcal{W}_I \cup \mathcal{Y} \cup \mathcal{W}'_J$. By the pairing theorem, $SFC(Y, \Gamma_I \cup \Gamma \cup \Gamma'_J) \simeq \widehat{BSD}(-\mathcal{W}_I) \boxtimes \widehat{BSAA}(\mathcal{Y}) \boxtimes \widehat{BSD}(\mathcal{W}'_J)$. But by Proposition 6.2, $\widehat{BSD}(-\mathcal{W}_I) = \{x_I, 0\}$ is the elementary module corresponding to ι_I , while $\widehat{BSD}(\mathcal{W}'_J) = \{y_J, 0\}$ is the elementary idempotent corresponding to ι'_J . Thus we have

$$\begin{aligned} \widehat{BSD}(-\mathcal{W}_I) \boxtimes \widehat{BSAA}(\mathcal{Y}) \boxtimes \widehat{BSD}(\mathcal{W}'_J) &= x_I \boxtimes \widehat{BSAA}(\mathcal{Y}) \boxtimes y_J \\ &\cong \iota_I \cdot \widehat{BSAA}(\mathcal{Y}) \cdot \iota'_J. \end{aligned}$$

Eq. (13) follows from Eq. (18) by substituting the empty sutured surface $\emptyset = (\emptyset, \emptyset)$ for \mathcal{F} . Its algebra is $\mathcal{A}(\emptyset) = \mathbb{Z}/2$, so ${}_{\mathbb{Z}/2}\widehat{BSAA}(\mathcal{Y})_B$ and $\widehat{BSA}(\mathcal{Y})_B$ can be identified.

Eq. (16) follows from Eq. (18) by substituting $\mathcal{F}(\mathcal{Z})$ for both \mathcal{F} and \mathcal{F}' , and $\mathcal{TW}_{-\overline{\mathcal{F}}, -}$ for \mathcal{Y} . Indeed, $\widehat{BSAA}(-\mathcal{TW}_{-\overline{\mathcal{F}}, -}) \simeq \mathcal{A}(\mathcal{Z})$, as a bimodule over itself, by Proposition 3.12.

Next we prove Eq. (17). Let U_A be a DG-type representative for $\widehat{BSA}(\mathcal{Y})_A$, and let M_I be the elementary representative for ${}_A\widehat{BSA}(\mathcal{W}_I)$. Since both are

DG-type, we can form the associative product

$$\begin{aligned} U \boxtimes {}^A\mathbb{I}^A \boxtimes M_I &\simeq \widehat{BSA}(Y) \boxtimes \widehat{BSD}(\mathcal{W}_I) \\ &\simeq SFC(Y, \Gamma \cup \Gamma_I). \end{aligned}$$

Similarly, pick M_J to be the elementary representative for $\widehat{{}_A BSA}(\mathcal{W}_J)$. We also know that ${}_A A_A$ is a DG-type representative for $\widehat{{}_A BSA}(\mathcal{TW}_{-\overline{\mathcal{F}}, -})_A$. We have the associative product

$$\begin{aligned} M_I^\vee \boxtimes {}^A\mathbb{I}^A \boxtimes A \boxtimes {}^A\mathbb{I}^A \boxtimes M_J &\simeq \widehat{BSD}(-\mathcal{W}_I) \boxtimes A \boxtimes \widehat{BSD}(\mathcal{W}_J) \\ &\simeq SFC(F \times [0, 1], \Gamma_{I \rightarrow J}). \end{aligned}$$

Gluing the two sutured manifolds along (F, Γ_I) results in

$$\mathcal{Y} \cup \mathcal{TW}_{\mathcal{F}, +} \cup \mathcal{TW}_{-\overline{\mathcal{F}}, -} \cup \mathcal{W}_J \cong \mathcal{Y} \cup \mathcal{W}_J = (Y, \Gamma \cup \Gamma_J),$$

so we get the correct manifold.

The gluing map can be written as the composition of

$$\begin{aligned} \Psi_{M_I}: (U \boxtimes \mathbb{I}) \boxtimes M_I \otimes M_I^\vee \boxtimes (\mathbb{I} \boxtimes A \boxtimes \mathbb{I} \boxtimes M_J) \\ \rightarrow (U \boxtimes \mathbb{I}) \boxtimes A^\vee \boxtimes (\mathbb{I} \boxtimes A \boxtimes \mathbb{I} \boxtimes M_J), \end{aligned}$$

$$\text{id}_U \boxtimes c_A \boxtimes \text{id}_{\mathbb{I} \boxtimes M_J}: U \boxtimes (\mathbb{I} \boxtimes A^\vee \boxtimes \mathbb{I} \boxtimes A) \boxtimes \mathbb{I} \boxtimes M_J \rightarrow U \boxtimes \mathbb{I} \boxtimes M_J,$$

where c_A is the homotopy equivalence from Lemma 5.6.

Luckily, since M_I is elementary, Ψ_{M_I} takes the simple form from Proposition 5.4. In addition, since U and M_J are DG-type, $\text{id} \boxtimes h \boxtimes \text{id}$ is also very simple. As can be seen in Figure 32, the composition is in fact

$$u \boxtimes * \boxtimes x_{I^c} \otimes x_{I^c}^\vee \boxtimes * \boxtimes a \boxtimes * \boxtimes x_{J^c} \mapsto m_{1|1}(u, a) \boxtimes * \boxtimes x_{J^c}.$$

Since $\cdot \boxtimes * \boxtimes x_{I^c}$ corresponds to $\cdot \iota_I$, this translates to the map

$$\begin{aligned} \Psi_{(F, \Gamma_I)}: (U \cdot \iota_I) \otimes (\iota_I \cdot A \cdot \iota_J) &\rightarrow U \cdot \iota_J, \\ (u \cdot \iota_I) \otimes (\iota_I \cdot a \cdot \iota_J) &\mapsto m(u, a) \cdot \iota_J \end{aligned}$$

Note that even though we picked a specific representative for $\widehat{BSA}(\mathcal{Y})_A$, the group $H_*(\widehat{BSA}(\mathcal{Y}))$ and the induced action m_H of $H_*(A)$ do not depend on this choice. Finally, Eq. (14) follows by treating A as a right module over itself. \square

APPENDIX A. CALCULUS OF DIAGRAMS

This appendix summarizes the principles of the diagrammatic calculus we have used throughout the paper. First we describe the algebraic objects we work with, and the necessary assumptions on them. Then we describe the diagrams representing these objects.

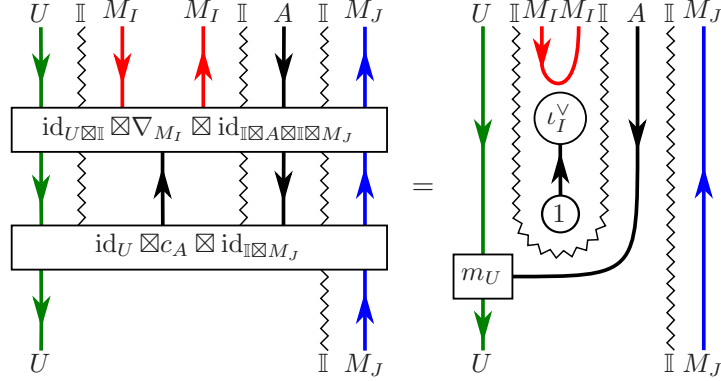


FIGURE 32. The gluing map Ψ_{M_I} on $SFC(Y, \Gamma_I) \otimes SFC(F \times [0, 1], \Gamma_{I \rightarrow J})$, followed by the chain homotopy equivalence $\text{id} \boxtimes c_A \boxtimes \text{id}$.

A.1. Ground rings. The two basic objects we work with are a special class of rings, and bimodules over them. We call these rings *ground rings*.

Definition A.1. A ground ring K is a finite dimensional $\mathbb{Z}/2$ -algebra with a distinguished basis (e_1, \dots, e_k) such that multiplication is given by the formula

$$e_i \cdot e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Such a basis for K is called a canonical basis.

The canonical basis elements are uniquely determined by the property that e_i cannot be written as a sum $u + v$, where u and v are nonzero and $u \cdot v = 0$. Each element of K is an idempotent, while $1_K = e_1 + \dots + e_k$ is an identity element.

We consider only finite dimensional bimodules ${}_K M_L$ over ground rings K and L , and collections $({}_K M_{iL})_{i \in I}$ where I is a countable index set (usually $I = \{0, 1, 2, \dots\}$, or some Cartesian power of the same), and each M_i is a finite-dimensional K, L -bimodule. It is often useful to think of the collection (M_i) as the direct sum $\bigoplus_{i \in I} M_i$, but that sometimes leads to problems, so we will not make this identification.

There are some basic properties of bimodules over ground rings as defined above.

Proposition A.2. Suppose K , L , and R are ground rings with canonical bases (e_1, \dots, e_k) , (e'_1, \dots, e'_l) , and (e''_1, \dots, e''_r) , respectively.

- A bimodule ${}_K M_L$ is uniquely determined by the collection of $\mathbb{Z}/2$ -vector spaces

$$e_i \cdot M \cdot e'_j, \quad i \in \{1, \dots, k\}, j \in \{1, \dots, l\},$$

which we will call the components of M .

- A K, L -bilinear map $f: M \rightarrow N$ is determined by the collection of $\mathbb{Z}/2$ -linear maps

$$f|_{e_i \cdot M \cdot e'_j}: e_i \cdot M \cdot e'_j \rightarrow e_i \cdot N \cdot e'_j.$$

- The tensor product $({}_K M_L) \otimes_L ({}_L N_R)$ has components

$$e_i \cdot (M \otimes_L N) \cdot e''_j = \bigoplus_{p=1}^l (e_i \cdot M \cdot e'_p) \otimes_{\mathbb{Z}/2} (e'_p \cdot N \cdot e''_j).$$

- The dual ${}_L M^\vee_K$ of ${}_K M_L$ has components

$$e_i \cdot M^\vee \cdot e'_j \cong (e'_j \cdot M \cdot e_i)^\vee,$$

and the double dual $(M^\vee)^\vee$ is canonically isomorphic to M .

Proof. These follow immediately. The fact that $M^{\vee\vee} \cong M$ is due to the fact the fact that each component is a finite dimensional vector space. \square

Finally, when dealing with countable collections we introduce the following conventions. For consistency we can think of a single module M as a collection (M_i) indexed by the set $I = \{1\}$.

Definition A.3. Let K , L , and M be as in Proposition A.2.

- An element of $(M_i)_{i \in I}$ is a collection $(m_i)_{i \in I}$ where $m_i \in M_i$.
- A bilinear map $f: ({}_K M_{iL})_{i \in I} \rightarrow ({}_K N_{jL})_{j \in J}$ is a collection

$$f_{(i,j)}: M_i \rightarrow N_j \quad (i,j) \in I \times J.$$

Equivalently, a map f is an element of the collection

$$\text{Hom}_{K,L}((M_i)_{i \in I}, (N_j)_{j \in J}) = (\text{Hom}(M_i, N_j))_{(i,j) \in I \times J}.$$

- The tensor ${}_K(M_i)_L \otimes_L (N_j)_R$ is the collection

$$((M \otimes N)_{(i,j)})_{(i,j) \in I \times J} = (M_i \otimes N_j)_{(i,j) \in I \times J}.$$

- The dual $((M_i)_{i \in I})^\vee$ is the collection $(M_i^\vee)_{i \in I}$.
- Given bilinear maps $f: (M_i) \rightarrow (N_j)$ and $g: (N_j) \rightarrow (P_p)$, their composition $g \circ f: (M_i) \rightarrow (P_p)$ is the collection

$$(g \circ f)_{(i,p)} = \sum_{j \in J} g_{(j,p)} \circ f_{(i,j)}.$$

Note that the composition of maps on collections may not always be defined due to a potentially infinite sum. On the other hand, the double dual $(M_i)^{\vee\vee}$ is still canonically isomorphic to (M_i) .

A.2. Diagrams for maps. We will use the following convention for our diagram calculus. There is a TQFT-like structure, where to decorated planar graphs we assign bimodule maps.

Proposition A.4. *Suppose $K_0, K_1, \dots, K_n = K_0$ are ground rings, $n \geq 0$, and ${}_{K_{i-1}}M_{iK_i}$ are bimodules, or collections of bimodules. Then the following $\mathbb{Z}/2$ -spaces are canonically isomorphic.*

$$\begin{aligned} A_i &= M_i \otimes M_{i+1} \otimes \dots \otimes M_n \otimes M_1 \otimes \dots \otimes M_{i-1} / \sim, \\ B_{i,j} &= \text{Hom}_{K_i, K_j}(M_i^\vee \otimes \dots \otimes M_1^\vee \otimes M_n^\vee \otimes \dots \otimes M_{j+1}^\vee, M_{i+1} \otimes \dots \otimes M_j), \\ C_{i,j} &= \text{Hom}_{K_j, K_i}(M_j^\vee \otimes \dots \otimes M_{i+1}^\vee, M_{j+1} \otimes \dots \otimes M_n \otimes M_1 \otimes \dots \otimes M_i), \end{aligned}$$

for $0 \leq i \leq j \leq n$, where the relation \sim in the definition of A_i is $k \cdot x \sim x \cdot k$, for $k \in K_{i-1}$.

Proof. The proof is straightforward. If all M_i are single modules, then we are only dealing with finite-dimensional $\mathbb{Z}/2$ -vector spaces. If some of them are collections, then the index sets for A_i , $B_{i,j}$ and $C_{i,j}$ are all the same, and any individual component still consists of finite dimensional vector spaces. \square

This property is usually referred to as *Frobenius duality*. Our bimodules behave similar to a pivotal tensor category. Of course we do not have a real category, as even compositions are not always defined.

Definition A.5. *A diagram is a planar oriented graph, embedded in a disc, with some degree-1 vertices on the boundary of the disc. There are labels as follows.*

- Each planar region (and thus each arc of the boundary) is labeled by a ground ring K .
- Each edge is labeled by a bimodule ${}_KM_L$, such that when traversing the edge in its direction, the region on the left is labeled by K , while the one on the right is labeled by L .
- An internal vertex with all outgoing edges labeled by M_1, \dots, M_n , in cyclic counterclockwise order, is labeled by an element of one of the isomorphic spaces in Proposition A.4.
- If any of the edges adjacent to a vertex are incoming, we replace the corresponding modules by their duals.

When drawing diagrams we will omit the bounding disc, and the boundary vertices. We will usually interpret diagrams consisting of a single internal vertex having several incoming edges M_1, \dots, M_m “on top”, and several outgoing edges N_1, \dots, N_n “on the bottom”, as a bilinear map in $\text{Hom}(M_1 \otimes \dots \otimes M_m, N_1 \otimes \dots \otimes N_n)$. See Figure 33 for an example.

Under some extra assumptions, discussed in Section A.3, a diagram with more vertices can also be evaluated, or interpreted as an element of some set, corresponding to all outgoing edges. The most common example is having two diagrams \mathcal{D}_1 and \mathcal{D}_2 representing linear maps

$$M \xrightarrow{f_1} N \xrightarrow{f_2} P.$$

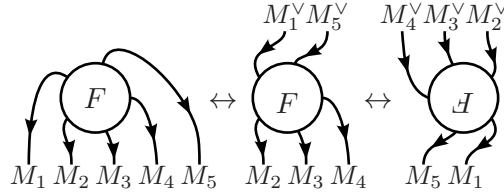


FIGURE 33. Three equivalent diagrams with a single vertex. The label F is interpreted as an element of $A_1 = M_1 \otimes \cdots \otimes M_5 / \sim$, $B_{1,4} = \text{Hom}(M_1^\vee \otimes M_5^\vee, M_2 \otimes M_3 \otimes M_4)$, and $C_{1,4} = \text{Hom}(M_4^\vee \otimes M_3^\vee \otimes M_2^\vee, M_5 \otimes M_1)$, respectively.

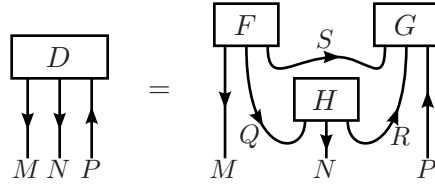


FIGURE 34. Evaluation of a complex diagram.

Stacking the two diagrams together, feeding the outgoing edges of \mathcal{D}_1 into the incoming edges of \mathcal{D}_2 , we get a new diagram \mathcal{D} , corresponding to the map $f_2 \circ f_1: M \rightarrow P$. More generally, we can “contract” along all internal edges, pairing the elements assigned to the two ends of an edge. As an example we will compute the diagram D in Figure 34. Suppose the values of the vertices F , G , and H are as follows:

$$\begin{aligned} F &= \sum_i m_i \otimes q_i \otimes s_i \in M \otimes Q \otimes S, \\ G &= \sum_j s'_j \otimes r'_j \otimes p'_j \in S^\vee \otimes R^\vee \otimes P^\vee, \\ H &= \sum_k q'_k \otimes n_k \otimes r_k \in Q^\vee \otimes N \otimes R. \end{aligned}$$

Then the value of D is given by

$$D = \sum_{i,j,k} \langle q_i, q'_k \rangle_Q \cdot \langle s_i, s'_j \rangle_S \cdot \langle r_k, r'_j \rangle_R \cdot m_i \otimes n_k \otimes p'_j \in M \otimes N \otimes P^\vee.$$

Edges that go from boundary to boundary and closed loops can be interpreted as having an identity vertex in the middle. As with individual vertices, we can rotate a diagram to interpret it as an element of different spaces, or different linear maps.

Note that the above construction might fail if any of the internal edges corresponds to a collection, since there might be an infinite sum involved. The next section discusses how to deal with this problem.

A.3. Boundedness. When using collections of modules we have to make additional assumptions to avoid infinite sums. We use the concept of boundedness of maps and diagrams.

Definition A.6. An element $(m_i)_{i \in I}$ of the collection $(M_i)_{i \in I}$ is called bounded if only finitely many of its components m_i are nonzero. Equivalently, the bounded elements of (M_i) can be identified with the elements of $\bigoplus_i M_i$.

For a collection $(M_{i,j})_{i \in I, j \in J}$ there are several different concepts of boundedness. An element $(m_{i,j})$ is totally bounded if it is bounded in the above sense, considering $I \times J$ as a single index-set. A weaker condition is that $(m_{i,j})$ is bounded in J relative to I . This means that for each $i \in I$, there are only finitely many $j \in J$, such that $m_{i,j}$ is nonzero. Similarly, an element can be bounded in I relative to J .

Note that $f: (M_i) \rightarrow (N_j)$ is bounded in J relative to I exactly when f represents a map from $\bigoplus_i M_i$ to $\bigoplus_j N_j$. In computations relatively bounded maps are more common than totally bounded ones. For instance the identity map $\text{id}: (M_i) \rightarrow (M_i)$ and the natural pairing $\langle \cdot, \cdot \rangle: (M_i)^\vee \otimes (M_i) \rightarrow K$ are not totally bounded, but are bounded in each index relative to the other.

To be able to collapse an edge labeled by a collection $(M_i)_{i \in I}$ in a diagram, at least one of the two adjacent vertices needs to be labeled by an element relatively bounded in the I -index. For a given diagram D we can ensure that it has a well-defined evaluation by imposing enough boundedness conditions on individual vertices. (There is usually no unique minimal set of conditions.) Total or relative boundedness of D can also be achieved by a stronger set of conditions. For example, if all vertices are totally bounded, the entire diagram is also totally bounded.

APPENDIX B. \mathcal{A}_∞ -ALGEBRAS AND MODULES

In this section we will present some of the background on \mathcal{A}_∞ -algebras and modules, and the way they are used in the bordered setting. A more thorough treatment is given in [LOT10a].

As in Appendix A, we always work with $\mathbb{Z}/2$ -coefficients which avoids dealing with signs. Everything is expressed in terms of the diagram calculus of Appendix A. As described there, all modules are finite dimensional, although we also deal with countable collections of such modules. There is essentially only one example of collections that we use, which is presented below.

B.1. The bar construction. Suppose K is a ground ring and ${}_K M_K$ is a bimodule over it.

Definition B.1. The bar of M is the collection

$$\text{Bar } M = (M^{\otimes i})_{i=0, \dots, \infty},$$

of tensor powers of M .

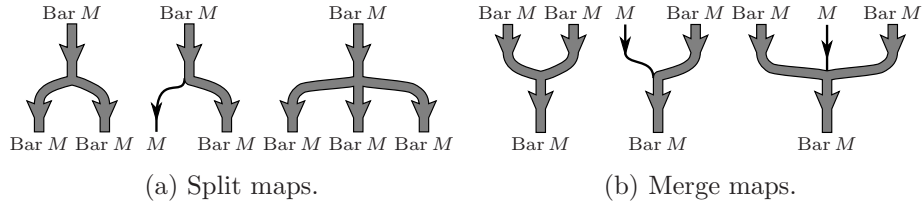


FIGURE 35

There are two important maps on the bar of M .

Definition B.2. *The split on $\text{Bar } M$ is the map $s: \text{Bar } M \rightarrow \text{Bar } M \otimes \text{Bar } M$ with components*

$$s_{(i,j,k)} = \begin{cases} \text{id}: M^{\otimes i} \rightarrow (M^{\otimes j}) \otimes (M^{\otimes k}) & \text{if } i = j + k, \\ 0 & \text{otherwise.} \end{cases}$$

The merge map $\text{Bar } M \otimes \text{Bar } M \rightarrow \text{Bar } M$ is similarly defined.

Merges and splits can be extended to more complicated situations where any combination of copies of $\text{Bar } M$ and M merge into $\text{Bar } M$, or split from $\text{Bar } M$. All merges are associative, and all splits are coassociative.

Like the identity map, splits and merges are bounded in incoming indices, relative to outgoing, and vice versa. To simplify diagrams, we draw merges and splits as merges and splits of arrows, respectively, without using a box for the corresponding vertex (see Figure 35).

B.2. Algebras and modules. The notion of an \mathcal{A}_∞ -algebra is a generalization of that of a differential graded (or DG) algebra. While the algebras that arise in the context of bordered Floer homology are only DG, we give the general definition for completeness. We will omit grading shifts.

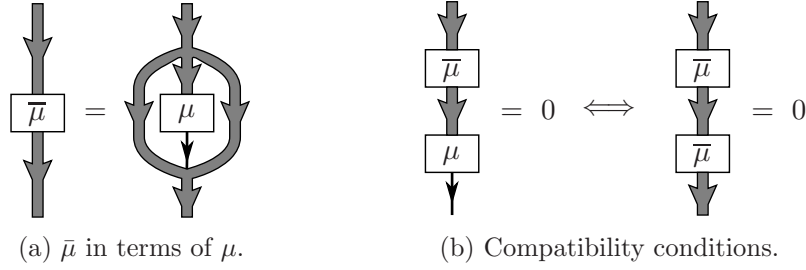
Definition B.3. *An \mathcal{A}_∞ -algebra A over the base ring K consists a K -bimodule ${}_K A_K$, together with a collection of linear maps $\mu_i: A^{\otimes i} \rightarrow A$, $i \geq 1$, satisfying certain compatibility conditions. By adding the trivial map $\mu_0 = 0: K \rightarrow A$, we can regard this as a map $\mu = (\mu_i): \text{Bar } A \rightarrow A$. This induces a map $\bar{\mu}: \text{Bar } A \rightarrow \text{Bar } A$, given by splitting $\text{Bar } M$ into three copies of itself, applying μ to the middle one, and merging again (see Figure 36a).*

The compatibility condition is $\bar{\mu} \circ \mu = 0$, or equivalently $\bar{\mu} \circ \bar{\mu} = 0$ (see Figure 36b).

The algebra is unital if there is a map $1: K \rightarrow A$ (which we draw as a circle labeled “1” with an outgoing arrow labeled “A”), such that $\mu_2(1, a) = \mu_2(a, 1) = a$, and $\mu_i(\dots, 1, \dots) = 0$ if $i \neq 2$.

The algebra A is bounded if μ is bounded, or equivalently if $\bar{\mu}$ is relatively bounded in both directions.

Notice that a DG-algebra with multiplication m and differential d is just an \mathcal{A}_∞ algebra with $\mu_1 = d$, $\mu_2 = m$, and $\mu_i = 0$ for $i \geq 3$. Moreover, DG-algebras are always bounded.

FIGURE 36. Definition of \mathcal{A}_∞ -algebras

Since DG-algebras are associative, there is one more operation that is specific to them.

Definition B.4. *The associative multiplication $\pi: \text{Bar } A \rightarrow A$ for a DG-algebra A is the map with components*

$$\pi_i(a_1 \otimes \cdots \otimes a_i) = \begin{cases} a_1 a_2 \cdots a_i & i > 0, \\ 1 & i = 0. \end{cases}$$

There are two types of modules: type- A , which is the usual notion of an \mathcal{A}_∞ -module, and type- D . There are four types of bimodules: type- AA , type- DA , etc. These can be extended to tri-modules and so on. We describe several of the bimodules. Other cases can be easily deduced.

Suppose A and B are unital \mathcal{A}_∞ -algebras with ground rings K and L , respectively. We use the following notation. A type- A module over A will have A as a lower index. A type- D module over A will have A as an upper index. Module structures over the ground rings K and L are denoted with the usual lower index notation.

Definition B.5. *A type- AA bimodule ${}_A M_B$ consists of a bimodule ${}_K M_L$ over the ground rings, together with a map $m = (m_{i|1|j}): \text{Bar } A \otimes M \otimes \text{Bar } B \rightarrow M$. The compatibility conditions for m are given in Figure 37.*

The bimodule M is *unital* if $m_{1|1|0}(1_A, m) = m_{0|1|1}(m, 1_B) = m$, and $m_{i|1|j}$ vanishes in all other cases where one of the inputs is 1_A or 1_B .

The bimodule can be *bounded*, *bounded only in A* , *relatively bounded in A with respect to B* , etc. These are defined in terms of the index sets of $\text{Bar } A$ and $\text{Bar } B$.

Definition B.6. *A type- DA bimodule ${}^A M_B$ consists of a bimodule ${}_K M_L$ over the ground rings, together with a map $\delta = (\delta_{i|1|j}): M \otimes \text{Bar } B \rightarrow A \otimes M$. This induces another map $\bar{\delta} = (\bar{\delta}_{i|1|j}): M \otimes \text{Bar } B \rightarrow \text{Bar } A \otimes M$, by splitting $\text{Bar } B$ into i copies, and applying δ i -many times (see Figure 38a). The compatibility conditions for δ and $\bar{\delta}$ are given in Figure 38b.*

The bimodule M is *unital* if $\delta_{1|1|1}(m, 1_B) = 1_A \otimes m$, and $\delta_{1|1|i}$ vanishes for $i > 1$ if one of the inputs is 1_B .

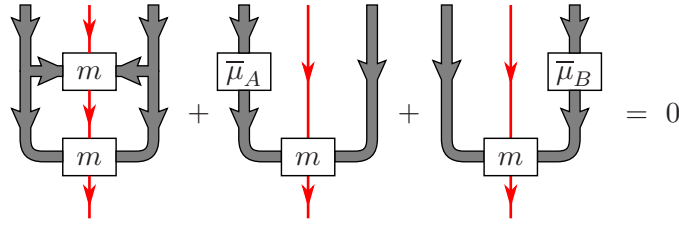
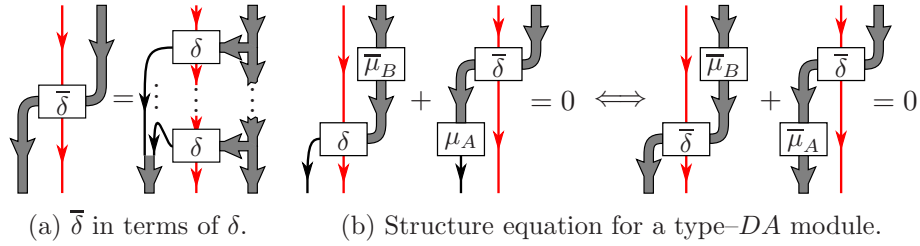
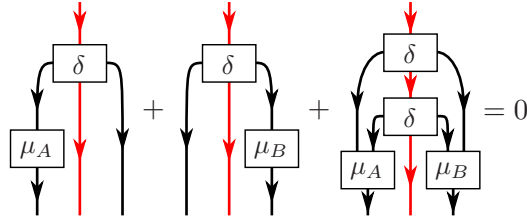
FIGURE 37. Structure equation for a type- AA module.(a) $\bar{\delta}$ in terms of δ .(b) Structure equation for a type- DA module.

FIGURE 38

FIGURE 39. Structure equation for a type- DD module.

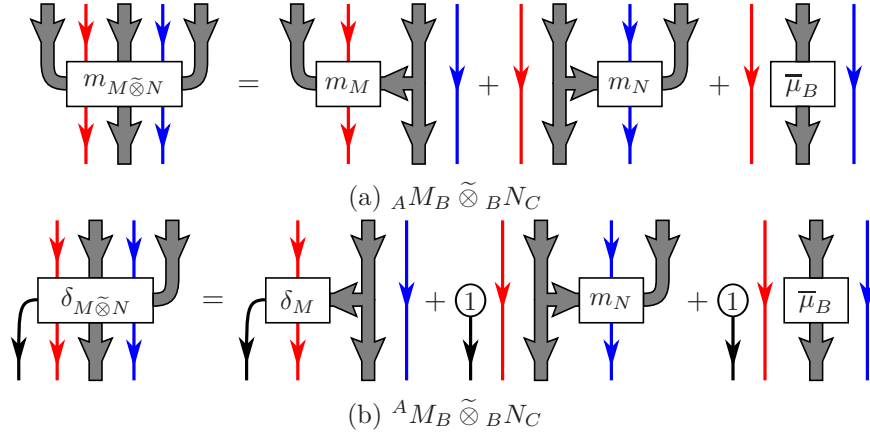
Again, there are various boundedness conditions that can be imposed.

Type- DD modules only behave well if the algebras involved are DG, so we only give the definition for that case.

Definition B.7. Suppose A and B are DG-algebras. A type DD -module ${}^A M^B$ consists of a bimodule ${}_K M_L$ over the ground rings, together with a map $\delta_{1|1|1}: M \rightarrow A \otimes M \otimes B$ satisfying the condition in Figure 39.

We omit the definition of one-sided type- A and type- D modules, as they can be regarded as special cases of bimodules. Type- A modules over A can be interpreted as type- AA bimodules over A and $B = \mathbb{Z}/2$. Similarly, type- D modules are type DA -modules over $\mathbb{Z}/2$.

B.3. Tensor products. There are two types of tensor products for \mathcal{A}_∞ -modules. One is the more traditional derived tensor product $\tilde{\otimes}$. It is generally hard to work with, as $M \tilde{\otimes} N$ is infinite dimensional over $\mathbb{Z}/2$ even when M and N are finite dimensional. This is bad for computational reasons, as well as when using diagrams—it violates some of the assumptions

FIGURE 40. Structure maps for two types of $\tilde{\otimes}$ products.

of Appendix A. Nevertheless, we do use it in a few places throughout the paper.

Throughout the rest of this section assume that A , B , and C are DG-algebras over the ground rings K , L , and P , respectively.

Definition B.8. Suppose ${}^A M_B$ and ${}^B N_C$ are two type-AA bimodules. The derived tensor product $({}^A M_B) \tilde{\otimes}_B ({}^B N_C)$ is a type-AA bimodule ${}^A (M \tilde{\otimes} N)_B$ defined as follows. Its underlying bimodule over the ground rings is

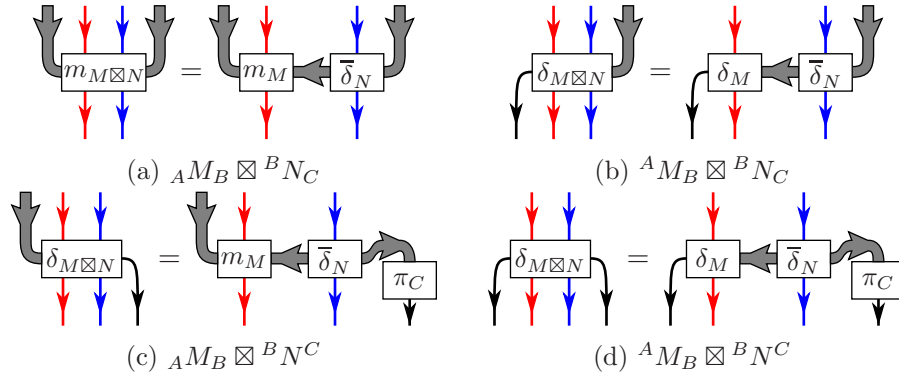
$$\begin{aligned} {}_K (M \tilde{\otimes} N)_P &= ({}_K M_L) \otimes_L \left(\bigoplus_{i=0}^{\infty} {}_L B_L^{\otimes i} \right) \otimes_L ({}_L N_P) \\ &= M \otimes_L \text{Bar } B \otimes_L N. \end{aligned}$$

Here we're slightly abusing notation in identifying $\text{Bar } B$ with a direct sum. The structure map as an \mathcal{A}_∞ -bimodule over A and C is $m_{M \tilde{\otimes} N}$, as shown in Figure 40a.

Similarly, we can take the derived tensor product of a DA module and an AA module, or a DA module and an AD module. The former is demonstrated in Figure 40b.

The other type of tensor product is the square tensor product \boxtimes . It is asymmetric, as it requires one side to be a type- D module, and the other to be a type- A module. The main advantage of \boxtimes over $\tilde{\otimes}$ is that $M \boxtimes N$ is finite dimensional over $\mathbb{Z}/2$ whenever M and N are. Its main disadvantage is that $M \boxtimes N$ is only defined subject to some boundedness conditions on M and N .

Definition B.9. Suppose ${}^A M_B$ is a type-AA bimodule and ${}^B N_C$ is a type- DA bimodule, such that at least one of M and N is relatively bounded in B . The square tensor product $({}^A M_B) \boxtimes_B ({}^B N_C)$ is a type-AA bimodule ${}^A (M \boxtimes N)_C$ defined as follows. Its underlying bimodule over the ground

FIGURE 41. Structure maps for the four types of \boxtimes products.

rings is

$$K(M \boxtimes N)_P = ({}_K M_L) \boxtimes_L ({}_L N_P),$$

and its structure map is $m_{M \boxtimes N}$ as shown in Figure 41a.

There are three other combinations depending on whether the modules are of type D or A with respect A and C . All combinations are shown in Figure 41.

B.4. Morphisms and homomorphisms. There are two different notions of morphisms when working with \mathcal{A}_∞ -modules and bimodules. The more natural one is that of *homomorphisms*, which generalize chain maps. However, if we work only with homomorphisms, too much information is lost. For this reason we also consider the more general *morphisms*. These generalize linear maps of chain complexes, which do not necessarily respect differentials.

Definition B.10. A morphism $f: M \rightarrow N$ between two bimodules M and N of the same type is a collection of maps of the same type as the structure maps for M and N . For example, $f: {}^A M_B \rightarrow {}^A N_B$ has components $f_{i|1|j}: \text{Bar } A \otimes M \otimes \text{Bar } B \rightarrow N$. The spaces of morphisms are denoted by ${}_A \text{Mor}_B(M, N)$, etc.

Suppose A and B are DG-algebras. The bimodules of each type, e.g. ${}_A \text{Mod}_B$, form a DG-category, with morphism spaces ${}_A \text{Mor}_B$, etc. The differentials and composition maps for each type are shown in Figures 42 and 43, respectively.

Definition B.11. A homomorphism $f: M \rightarrow N$ of bimodules is a morphism f which is a cycle, i.e., $\partial f = 0$. A null-homotopy of f is a morphism H , such that $\partial H = f$. The space of homomorphisms up to homotopy is denoted by ${}_A \text{Hom}_B$, etc.

Notice that the homomorphism space ${}_A \text{Hom}_B(M, N)$ is exactly the homology of ${}_A \text{Mor}_B(M, N)$. This gives us a new category of bimodules.

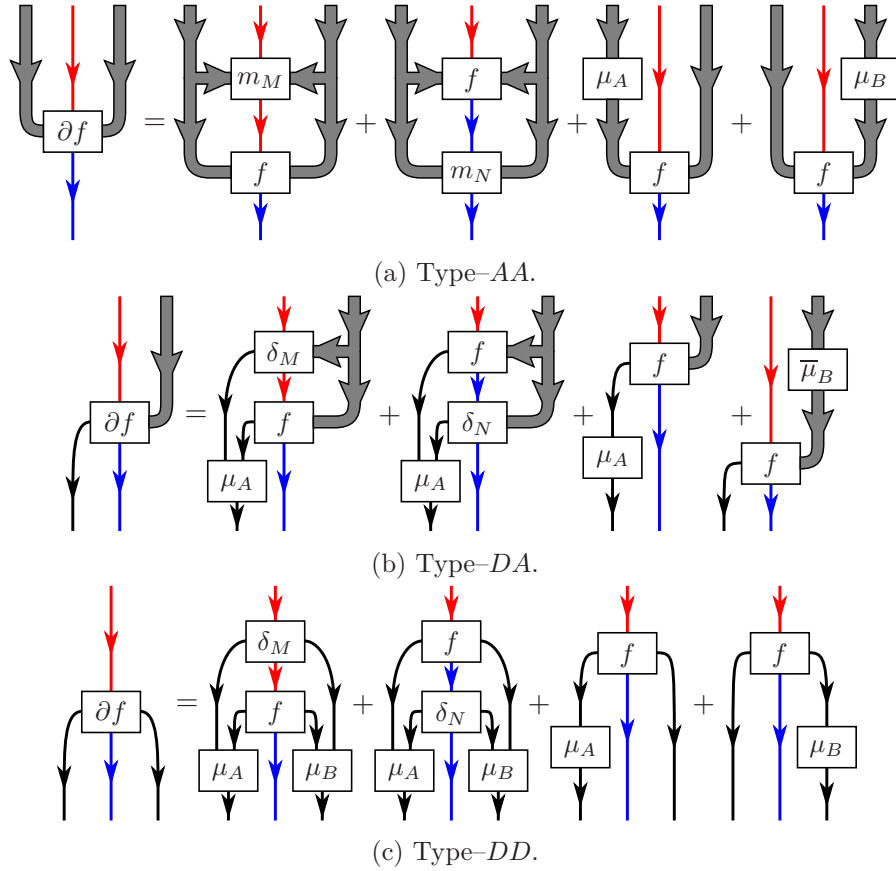


FIGURE 42. Differentials of the different types of morphisms.

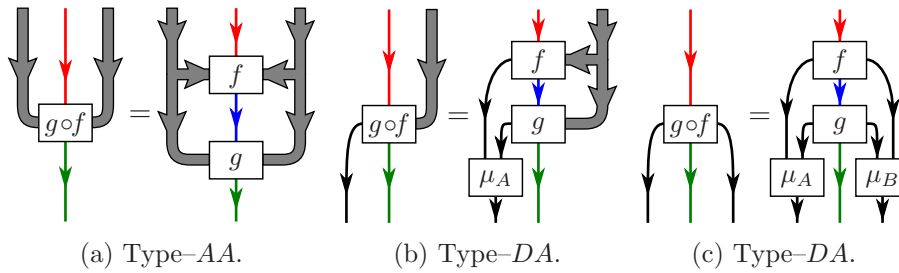


FIGURE 43. Compositions of the different types of morphisms.

Having homomorphisms and homotopies allows us to talk about homotopy equivalences of modules. For example, if ${}_A M_B$ is a bimodule, then $A \tilde{\otimes} M \simeq M \simeq M \tilde{\otimes} B$, via canonical homotopy equivalences. For example, there is $h_M: A \tilde{\otimes} M \rightarrow M$, which we used in several places.

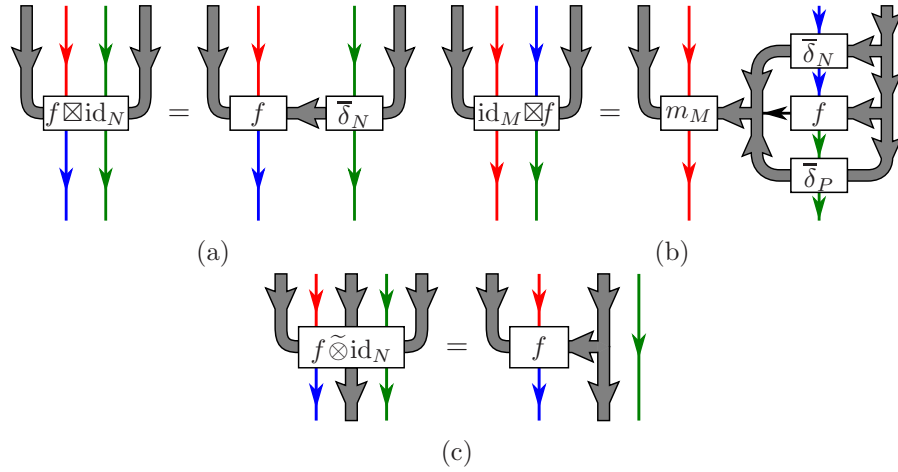
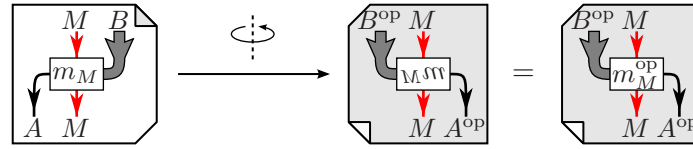


FIGURE 44. Three types of induced maps on tensor products.

FIGURE 45. Passing from ${}^A\text{Mod}_B$ to ${}_{B^{\text{op}}}\text{Mod}^{A^{\text{op}}}$ by reflection.

B.5. Induced morphisms. Suppose $f: M \rightarrow N$ is a bimodule morphism. This induces morphisms

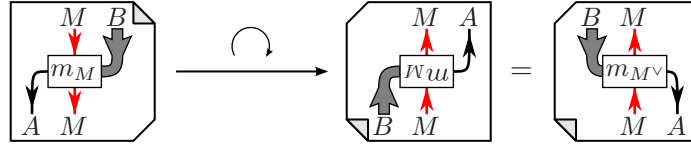
$$f \tilde{\otimes} \text{id}: M \tilde{\otimes} P \rightarrow N \tilde{\otimes} P \quad f \boxtimes \text{id}: M \boxtimes P \rightarrow N \boxtimes P,$$

whenever the tensor products are defined. The main types of induced morphisms are shown in Figure 44. The functors $\cdot \boxtimes \text{id}$ and $\cdot \tilde{\otimes} \text{id}$ are DG-functors. That is, they preserve homomorphisms, homotopies, and compositions.

B.6. Duals. There are two operations on modules, which can be neatly expressed by diagrams. One is the operation of turning a bimodule ${}_A M_B$ into a bimodule ${}_{B^{\text{op}}} M_{A^{\text{op}}}$. (Similarly, type- DA bimodules become type- AD bimodules, etc.) Diagrammatically this is achieved by reflecting diagrams along the vertical axis. See Figure 45 for an example.

The other operation is dualizing modules and bimodules. If ${}_A M_B$ has an underlying bimodule ${}_K M_L$ over the ground rings, then its dual ${}_B M_A^{\vee}$ has an underlying bimodule ${}_L M_K^{\vee} = ({}_K M_L)^{\vee}$. Diagrammatically this is achieved by rotating diagrams by 180 degrees. Again, there are variations for type- D modules. See Figure 46 for an example.

Since the structure equations are symmetric, it is immediate that both of these operations send bimodules to bimodules, as long as we restrict to modules finitely generated over $\mathbb{Z}/2$.

FIGURE 46. Passing from ${}^A\text{Mod}_B$ to ${}_B\text{Mod}^A$ by rotation.

This gives equivalences of the DG-categories

$${}^A\text{Mod}_B \cong {}_{B^{\text{op}}}\text{Mod}^{A^{\text{op}}} \cong ({}_B\text{Mod}^A)^{\text{op}},$$

etc. One can check that both constructions extend to tensors, induced morphisms, etc.

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DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027
E-mail address: rzarev@math.columbia.edu